Comparison Theorems in Algebraic and Analytic Geometry

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Abstract

This master thesis develops a general method for lifting classical identifications of cohomology groups to equivalences between the cohomology of ∞-topoi, yielding analogues of the comparison theorems of Artin [AGVb, XVI.4.1] and Berkovich [Ber93, 7.5.1] for sheaves valued in spaces. Functoriality is established via a proper base change theorem for space-valued sheaves on Berkovich analytic spaces, following the approach of [Cho22]. As a consequence, the profinite étale homotopy type of a scheme is identified with the homotopy type of its analytification. Finally, the same method applies to the cdh topology, showing that over fields admitting resolution of singularities the profinite cdh-homotopy type of a scheme is constant and determined by the homotopy type of its analytification, generalizing comparison results for cohomology groups due to [MV24].

To Elli, my parents, and their constant support and saintly patience.

Contents

1 Introduction

The classical Comparison Theorem of Artin [AGVb, XVI.4.1] establishes a robust bridge between algebraic geometry and complex analytic geometry. Specifically, every morphism $f\colon X \to S$ of schemes locally of finite type over $\mathbb{C}$ induces a square of categories of abelian sheaves

$$\begin{tikzcd}
\mathrm{Shv}_{\acute{e}t}(X, \mathrm{Ab}) \arrow[r, "f_*"] \arrow[d, "(\text{-})^{\mathrm{an}}"] & \mathrm{Shv}_{\acute{e}t}(S, \mathrm{Ab}) \arrow[d, "(\text{-})^{\mathrm{an}}"] \\
\mathrm{Shv}_{\mathrm{open}}(X(\mathbb{C})) \arrow[r, "f_*^{\mathrm{an}}"] & \mathrm{Shv}_{\mathrm{open}}(S(\mathbb{C}))
\end{tikzcd}$$

If f is a proper morphism or $F \in \mathrm{Shv}_{\acute{e}t}(X, \mathrm{Ab})$ is a constructible sheaf, this square gives rise to a quasi-isomorphism

$$(Rf_*F)^{\mathrm{an}} \xrightarrow{\simeq} Rf_*^{\mathrm{an}} F^{\mathrm{an}}$$

demonstrating that the square commutes up to homotopy-equivalence, when passing to the derived categories of chain complexes. In case $S = \mathrm{Spec}(\mathbb{C})$ this identifies the étale cohomology of X (with constructible coefficients) with the singular cohomology the of the underlying complex analytic space of $X(\mathbb{C})$.

Artin and Mazur [AM69] subsequently introduced the *étale homotopy type* of a scheme X, a (pro-)object in the homotopy category of topological spaces. This invariant recovers the étale cohomology and fundamental group of X, and relates to the homotopy type of $X(\mathbb{C})$ via *profinite completion*, a notion motivated by the profinite completion of groups.

Recent developments in higher topos theory [HTT] – particularly the results of [Car16; Cho22] – allow us to extend these comparison theorems to sheaves of spaces (higher stacks), demonstrating that this correspondence is intrinsic to the geometry, independent of any choice of coefficients. By employing Berkovich's theory of analytic spaces [Ber93] – a natural generalization of the complex analytic setting – we establish these general comparison theorems simultaneously over non-archimedean fields.

Berkovich analytic spaces constitute a robust topological framework: They are locally Hausdorff and locally compact, and admit well-behaved notions of étale cohomology and motives (cf. [Sch25]). Despite these advantages, the foundational literature for this theory remains less exhaustive than that of rigid or algebraic geometry. Consequently, it can be unclear whether standard algebraic results possess Berkovich analogues. For instance, the proper base change strategy employed in [Cho22] requires the étale topos to have *enough points*, all of which are induced by geometric points. As this result appears to be absent from the literature, we provide a proof in Chapter 7

To ensure this work is self-contained and accessible to readers with a background in classical algebraic geometry, we provide concise introductions to the theory of ∞-topoi, as well as complex and Berkovich analytic geometry

Preliminaries 1.0.1.

- Especially in sections 2– 4, we assume foundational knowledge in the language of ∞-categories – say of the level of [Lan21]. Whenever, topics are not contained in [Lan21], we freely use the language developed in [HTT], supplemented with [nLab] where suitable.
  For the student familiar with [Lan21] we include a brief guide to functoriality of the ∞-categorical slice construction, tensors over ∞-groupoids and internal mapping spaces providing a smooth transition to the topics presented here and minimizing the dependencies on [HTT].
- We suppose the reader is familiar with basic concepts in homological algebra, like spectral sequences and derived categories in general, say at the level of [Wei94].
- We assume a fundamental background in algebraic geometry, say [Har77, I-III] partially supplemented with [FOAG]. Cohomology and derived functors will be treated in perspective of [Wei94] and ∞-categories respectively.
- Familiarity with basic concepts of algebraic topology, like Eilenberg-MacLane spaces, will be inevitable. [Die10] is more than sufficient.

As conventions and notation in the sources listed in the preliminaries above are not uniform, we fix a notation for some key concepts for convenience for the reader.

Notation 1.0.2.

- For a ∞-category $\mathcal{C}$, $\mathcal{P}(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ani})$ is the category of presheaves.
- $*$ stands for "the point" in whatever context suitable. For example in the context of ∞-category $*$ is the simplicial set Δ^0, while in the context of sets $*$ denotes the one point set.
- The Yoneda embedding for an ∞-category (see e.g. [Lan21, §4.2]) and the opposite Yoneda are embedding denoted by

$$y \colon \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ani}) \qquad\qquad y^{\mathrm{op}} \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{C}, \mathrm{Ani}).$$

- qcqs is short for quasi-compact quasi-separated.
- Given an abelian category A, we denote its derived category by $\mathrm{D}(A)$. If A is the category of sheaves on a site on a space X, we often abbreviate $\mathrm{D}(X) := \mathrm{D}(\mathrm{Shv}(X; \mathrm{Ab})$.
- A functor of (∞-)categories is left exact if it preserves finite limits. Likewise, right exact is synonymous to finite colimit preserving
- In the context of ∞-categories, unique means essentially unique, i.e. unique up to contractible choice.
- Given a functor of ∞-categories $F \colon \mathcal{C} \to \mathcal{D}$, a morphism $c \to c'$ in $\mathcal{C}$ is a F-equivalence, if it $F(c) \to F(c')$ is

Lastly, we advert a uniform notation for the universal homotopy category and its subcategories, which historically has had many names

Notation 1.0.3. Denote by An the ∞-category of anima – that is, of ∞-groupoids, Kan complexes, or equivalently CW-complexes. The following notations will be used for its full subcategories.

1) $i_{<\infty}\colon \mathrm{Ani}^{<\infty} \hookrightarrow \mathrm{Ani}$ is the full subcategory of truncated anima, i.e. every $K \in \mathrm{Ani}^{<\infty}$ only has finitely many non-zero homotopy groups.
2) $i_\pi\colon \mathrm{An}^\pi \hookrightarrow \mathrm{Ani}^{<\infty}$ is full subcategory of **π-finite** anima i.e. all $K \in \mathrm{Ani}^\pi$ have finitely many connected components, finitely many non-zero homotopy groups and all homotopy groups of K are finitely generated.
3) $i_{\mathrm{ab}}\colon \mathrm{Ani}^{\mathrm{ab}} \hookrightarrow \mathrm{Ani}$ is the full subcategory of anima K, whose fundamental group $\pi_1(K)$ is abelian.
4) Let p be a prime. Let $i^p\colon \mathrm{Ani}^{p\text{-free}} \hookrightarrow \mathrm{Ani}^\pi$ denote the full subcategory of **p-coprime** anima i.e. those $K\mathrm{Ani}$ such that each $\pi_n(K)$ is p-torsion free i.e. the order of $\pi_n(K)$ is not divisible by p for each $n \geq 1$.

2 Recollections on ∞-Topoi

2.1 The categories $\mathcal{RT}\mathrm{op}$ and $\mathcal{LT}\mathrm{op}$

On this first page, the fundamental terminologies around ∞-topoi is established, as this is necessary for a more detailed discussion in the following subsections.

Definition 2.1.1 (∞-Topos)**.** An ∞-category $\mathcal{X}$ is an **∞-topos**, if there exists a small ∞-category $\mathcal{C}$ and an accessible, left exact localization $L_{\mathcal{X}} \colon \mathcal{P}(\mathcal{C}) \to \mathcal{X}$.

Note. What is simply denoted here as localization in terms of [HTT], is also referred to as (left) Bousfield localization e.g in [Lan21] and [Lur18].

Remark 2.1.2. Every ∞-topos is a presentable ∞-category, since – as shown by Simpson [Sim99] – presentable categories are precisely those ∞-categories $\mathcal{D}$, which admit an accessible localization $\mathcal{P}(\mathcal{C}) \to \mathcal{D}$ from the presheaves on a small ∞-category $\mathcal{C}$ (see also [HTT, 5.5.1.1 (5)]). The additional structure of an ∞-topos endows it with several further desirable properties. In particular, all colimits in an ∞-topos are universal[1] by [HTT, 6.1.3.10].

Similar to ordinary topos theory, the natural notion of morphisms between ∞-topoi consists of an adjoint pair similar to $L_{\mathcal{X}} \dashv i_{\mathcal{X}}$ of Def. 2.1.1.

Definition 2.1.3. A map $f_* : \mathcal{X} \to \mathcal{Y}$ of ∞-topoi is **geometric** if it admits a left exact left adjoint f^*. Usually, this situation is displayed as $\mathcal{X} \overset{f^*}{\underset{f_*}{\leftrightarrows}} \mathcal{Y}$.
Denote by $\mathcal{RT}\mathrm{op}$ the subcategory[2] of $\widehat{\mathrm{Cat}}_\infty$, spanned by ∞-topoi and their geometric morphisms. Correspondingly, let $\mathcal{LT}\mathrm{op}$ be the subcategory of $\widehat{\mathrm{Cat}}_\infty$, spanned by ∞-topoi and the left exact left adjoints of geometric morphisms. By abuse of notation, the arrows of $\mathcal{LT}\mathrm{op}$ are called geometric morphisms, as well.
For ∞-topoi $\mathcal{X}$ and $\mathcal{Y}$, let $\mathrm{Fun}^*(\mathcal{X}, \mathcal{Y}) \subset \mathrm{Fun}(\mathcal{X}, \mathcal{Y})$ denote the full subcategory spanned by morphisms of $\mathcal{LT}\mathrm{op}$. If the context allows, this notation is extended to usual ∞-categories and left-exact, colimit preserving functors.

It readily follows, that these two subcategories of $\widehat{\mathrm{Cat}}_\infty$ are two sides of a single coin.

Proposition 2.1.4 ([HTT, 6.3.1.8])**.** *The categories* $\mathcal{LT}\mathrm{op}$ *and* $\mathcal{RT}\mathrm{op}$ *are anti-equivalent in the sense, that for every* $S \in \mathbf{Set}^{\Delta^{\mathrm{op}}}$*, there is a canonical equivalence*

$$\mathrm{Fun}(S, \mathcal{LT}\mathrm{op}) \simeq \mathrm{Fun}(S^{\mathrm{op}}, \mathcal{RT}\mathrm{op})$$

The categories of ∞-topoi are reasonably complete, although their (co)limits do not in general coincide with those of $\widehat{\mathrm{Cat}}_\infty$.

Proposition 2.1.5 ([HTT, 6.3.2.3], [HTT, 6.3.3.1])**.** *The category* $\mathcal{RT}\mathrm{op}$ *admits small colimits and small cofiltered limits. The inclusion* $\mathcal{RT}\mathrm{op} \subset \widehat{\mathrm{Cat}}_\infty$ *preserves small cofiltered limits, but does not preserve colimits in general. The inclusion* $\mathcal{LT}\mathrm{op} \subset \widehat{\mathrm{Cat}}_\infty$ *preserves small limits.*

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[1] See Def. A.2.4.

[2] The collection of geometric morphisms is closed under equivalences, since adjunctions are closed under composition and every equivalence is (left and right) adjoint to its inverse.

2.2 Grothendieck sites and ∞-sheaves

Although, the explicit theory of ∞-topoi is rather young, implicitly this notion has played an essential role in mathematics all along, beginning with the rise of sheaf theory and simplicial methods. This is because, not only all presheaves, but also all sheaves with values in a presentable category are ∞-topoi. Since many important examples stem from this context, the transition from the usual setup of Grothendieck topologies to ∞-sites and ∞-topoi will be discussed in detail here. The theory presented here is not entirely standard, as this section incorporates and extends Grothendieck *pretopologies*, which generalize the notion found in [Mil13, §5] or [KS06, 16.1.5] and first appeared [Pst22, A.1], explicitly. To ensure these pretopologies serve as a basis for usual Grothendieck topologies, as presented in [HTT, §6.2.2], this usual framework is introduced first.

Definition 2.2.1 (Sieves)**.** A **sieve** on an ∞-category $\mathcal{C}$ is a full subcategory $S \subset \mathcal{C}$, which is closed under precomposition i.e if $x \to y$ is a map in $\mathcal{C}$ and $y \in S$, then $x \in S$. A sieve on an object $x \in \mathcal{C}$ is a sieve on the slice category $\mathcal{C}_{/y}$. The collection of all sieves on $\mathcal{C}$ forms a poset, denoted by $\mathrm{Sieve}(\mathcal{C})$.

Definition 2.2.2. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and $S \subset \mathcal{D}$ a sieve. The pullback

$$\begin{tikzcd} F^{-1}S \arrow[r] \arrow[d] & S \arrow[d] \\ \mathcal{C} \arrow[r, "F"] & \mathcal{D} \end{tikzcd}$$

is a full subcategory of $\mathcal{C}$, commonly refer to as **pullback sieve** of S along F.

Observation 2.2.3. By its definition as a pullback, it is evident that the pullback sieve $F^{-1}S$ is a sieve. If F is the functor $f_!\colon \mathcal{C}_{/x} \to \mathcal{C}_{/y}$ induced by a morphism $f\colon x \to y$ and $S_y \subset \mathcal{C}_{/y}$ is a sieve, then $f_!^{-1}S_y$ is the full subcategory spanned by the $g\colon z \to x$ in $\mathcal{C}_{/x}$ for which $f \circ g \in S_y$.

Definition 2.2.4. A **Grothendieck topology** τ on an ∞-category $\mathcal{C}$ is an assignment of a full sub-1-category $\mathrm{CovS}_\tau(x) \subset \mathrm{Sieve}(\mathcal{C}_{/x})$ to every $x \in \mathcal{C}$. The elements of $\mathrm{CovS}_\tau(x)$ are called **covering sieves** and must satisfy the following properties:

1) For any $x \in \mathcal{C}$, the whole slice category $\mathcal{C}_{/x}$ is in $\mathrm{CovS}_\tau(x)$.
2) Let $f : y \to x$ be a morphism in $\mathcal{C}$ and $S_x \in \mathrm{CovS}_\tau(x)$ a covering sieve. Let $f_!^{-1}S_x$ be the pullback sieve along the forgetful map $f_!$. Then, $f_!^{-1}S_x \in \mathrm{CovS}_\tau(y)$.
3) Let $S_x \subset \mathcal{C}_{/x}$ be some sieve and let $S'_x \subset \mathcal{C}_{/x}$ be a covering sieve on x. Suppose that $f_!^{-1}S_x \in \mathrm{CovS}_\tau(y)$ for all objects $f : y \to x$ in S'_x. Then, $S_x \in \mathrm{CovS}_\tau(x)$.

As usual, a **site** is an ∞-category equipped with a Grothendieck topology.

Defining sites directly from the definition is often cumbersome, similar to ordinary Grothendieck topologies. Consequently, in practice, sites are frequently defined by implicitly specifying a generating set. This procedure can be made explicit in the following way.

Definition 2.2.5 ([Pst22, A.1])**.** A **Grothendieck pretopology** τ on an ∞-category $\mathcal{C}$ is an assignment of a collection $\mathrm{Cov}_\tau(c)$ of families of maps $\{c_i \to c\}_{i\in I}$ to each $c \in \mathcal{C}$, called **coverings** of c. Those families are required to satisfy the following properties:

1) For every equivalence $d \to c$, the set $\{d \to c\}$ is in $\mathrm{Cov}_\tau(c)$.

2) For any map $d \to c$ in C and any $\{c_i \to c\}_{i\in I} \in \mathrm{Cov}_\tau(c)$, the pullbacks $d \times_c c_i$ exist in C and the class $\{d \times_c c_i \to d\}_{i \in I}$ is a covering of d.

3) Let $\{c_i \to c\}_{i\in I} \in \mathrm{Cov}_\tau(c)$ and $\{c_{i,j} \to c_i\}_{j \in J_i} \in \mathrm{Cov}_\tau(c_i)$ for every $i \in I$. Then, the family of composites $\{c_{i,j} \to c\}_{i\in I, j \in J_i}$ is a covering of c.

An ∞-category $\mathcal{C}$ equipped with a Grothendieck pretopology τ is called a **presite**.

Observation 2.2.6. Def. 2.2.5 directly generalizes the Definition of ordinary Grothendieck pretopologies, Indeed, a collection of coverings τ on an ∞-category $\mathcal{C}$ is a presite if and only if $(h\mathcal{C}, \tau)$ forms a presite in the sense of [Mil13, §5] or [KS06, 16.1.5]. In particular, a 1-category C forms a presite (C, τ) if and only if $(N(C), \tau)$ is an ∞-presite.

Remark 2.2.7. The Observation above holds for Grothendieck topologies analogously. According to [HTT, 6.2.2.3], specifying Grothendieck topology on a ∞-category $\mathcal{C}$ amounts to giving an ordinary Grothendieck topology[3] on $h\mathcal{C}$.

The next Lemma verifies, that every pretopology generates a topology, as usual.

Lemma 2.2.8. *Let τ be a pretopology on an ∞-category $\mathcal{C}$. Define a sieve $S_x \subset \mathcal{C}_{/x}$ over $x \in \mathcal{C}$ to be a covering sieve, if it contains a covering $\{c_i \to c\} \in \mathrm{Cov}_\tau(c)$. The collection of all such covering sieves forms a Grothendieck topology.*

Proof. Let us verify the conditions of Def. 2.2.4 for the collection of covering sieves defined above.

1) For every $x \in \mathcal{C}$ the whole category $\mathcal{C}_{/x}$ is a covering sieve, since it contains the covering $\{x \xrightarrow{=} x\}$.

2) Let $S_x \subset \mathcal{C}_{/x}$ be a covering sieve containing a covering $\{x_i \to x\}_{i\in I}$. Then, for every morphism $f\colon y \to x$, the pullback sieve $f_!^{-1} S_x$ contains the covering $\{y \times_x x_i \to y\}_{i \in I}$, thus is a covering sieve of y.

3) Let S_x be a sieve on x, which contains a covering $\{x_i \xrightarrow{f_i} x\}_{i\in I} \in \mathrm{Cov}_\tau(x)$. Let S'_x be another sieve on x. Suppose, that for each $i \in I$ the pullback sieve $(f_i)_!^{-1}\, \mathcal{C}^1_{/x}$ again contains a covering $\{x_{i,j} \to x_i\}_{j \in J_i}$. Then, $\{x_{i,j} \to x\}_{i \in I, j\in J_i}$ forms a covering of x by Def. 2.2.5. The description of $(f_i)_!^{-1}\, S'_x$ in Obs. 2.2.3 shows that each map $x_{i,j} \to x$ is in S'_x. In consequence, S'_x contains the covering $\{x_{i,j} \to x\}_{i,j}$. ☐

Notation 2.2.9. Let $(\mathcal{C}, \tau)$ be a site, $x \in \mathcal{C}$ and let $S \subset \mathcal{C}_{/x}$ be a sieve on x. In view of Lem. 2.2.8, S is said to be generated by a family $\{x_i \to x\}_{i \in I} \subset S$ if S is the smallest sieve on x containing every morphism $x_i \to x$ for $i \in I$.

Definition 2.2.10. Let $(\mathcal{C}, \tau)$ be a site and $\mathcal{D}$ a complete ∞-category. Let $x \in \mathcal{C}$ be an object and $S_x \in \mathrm{CovS}_\tau(x)$ a covering sieve for x.

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[3] The Definition of Grothendieck topology for 1-categories given in [MM12] readily generalizes to Def. 2.2.4 because of [HTT, 6.2.2.9].

A functor $F : \mathcal{C}^{op} \to \mathcal{D}$ is said to be $\mathbf{S_x}$**-local**, if $S_x \subset \mathcal{C}_{/x} \to \mathcal{C}$ induces an equivalence

$$F(x) \simeq \lim_{y \in S_x} F(y).$$

Moreover, an $F \colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ is a **sheaf** on $\mathcal{C}$ with values in $\mathcal{D}$, if it is S_x-local for every $x \in \mathcal{C}$ and every covering sieve $S_x \in \mathrm{CovS}_\tau(x)$. Let $\mathrm{Shv}(\mathcal{C}; \mathcal{D}) \subset \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$ denote the full subcategory of sheaves on $\mathcal{C}$ with values in $\mathcal{D}$. If $\mathcal{D} = \mathrm{An}$, then $\mathrm{Shv}(\mathcal{C}; \mathrm{An})$ is abbreviated by $\mathrm{Shv}(\mathcal{C})$ and one refers to the objects of $\mathrm{Shv}(\mathcal{C})$ as sheaves on $\mathcal{C}$.

Notation 2.2.11. Moreover, if $(\mathcal{C}, \tau)$ is a presite, let $\mathrm{Shv}_\tau(\mathcal{C})$ denote the category of sheaves on the site associated to $\mathcal{C}$. In view of Obs. 2.2.6 and Rmk 2.2.7, one writes $\mathrm{Shv}_\tau(C)$ instead of $\mathrm{Shv}_\tau(N(C))$, for a 1-(pre)site (C, τ).

Warning 2.2.12. Lem. 2.11.5 will show, that if $\mathcal{D}$ is an ∞-topos, then $\mathrm{Shv}_\tau(\mathcal{C}; \mathcal{D})$ is an ∞-topos, as well. This is false in general.

Definition 2.2.13. Let $\mathcal{X}$ be a ∞-topos and let $\mathcal{X} \overset{i}{\subset} \mathcal{P}(\mathcal{C})$ be the fully faithful right adjoint to its defining localization. A morphism $X \to Y$ in $\mathcal{X}$ is a **monomorphism** if, for every $x \in \mathcal{C}$, the map $i(X)(x) \to i(Y)(x)$ has solely empty or contractible fibers.

Construction 2.2.14. Let $\mathcal{C}$ be a small ∞-category, $x \in \mathcal{C}$ and let $U \to y(x)$ be a monomorphism in $\mathcal{P}(\mathcal{C})$. Let S_U denote the pullback

$$\begin{tikzcd}
S_U \arrow[r] \arrow[d] & \mathcal{C}_{/x} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\
\mathcal{P}(\mathcal{C})_{/U} \arrow[r] & \mathcal{P}(\mathcal{C})_{/y(x)} \arrow[r] & \mathcal{P}(\mathcal{C})
\end{tikzcd}$$

then, S_U is the sieve on x given by the full subcategory of $\mathcal{C}_{/x}$ spanned by those $c \in \mathcal{C}_{/x}$, such that $y(c) \to y(x)$ factors through U.

Remark 2.2.15. As shown in [HTT, 6.2.2.4], Construction 2.2.14 determines a bijection for every $x \in \mathcal{C}$

$$\{\text{sieves over } x\} \xleftrightarrow{1:1} \{\text{monomorphisms } U \hookrightarrow y(x)\}$$

Most importantly, for a site $(\mathcal{C}, \tau)$ and a covering sieve $S_x \in \mathrm{CovS}_\tau(x)$ it is shown in [HTT, 6.2.2.5], that a presheaf F is S_x-local if and only if F is U_{S_x}-local
(i.e. $F(x) \cong \mathrm{map}_{\mathcal{P}(\mathcal{C})}(U_{S_x}, F)$), where $U_{S_x} \to y(x)$ is the monomorphism associated to S_x. Let S_τ denote the class of monomorphisms corresponding to the covering sieves of τ. Then, the saturated closure $\bar{S_\tau}$ is is stable under pullback by [HTT, 6.2.2.7]. In consequence, $\mathrm{Shv}_\tau(\mathcal{C})$ is the localization $\mathcal{P}(\mathcal{C})[S_\tau^{-1}]$ and particular an ∞-topos, since
$L \colon \mathcal{P}(\mathcal{C}) \xrightarrow{S_\tau^{-1}} \mathcal{P}(\mathcal{C})[S_\tau^{-1}] \simeq \mathrm{Shv}_\tau(\mathcal{C})$ admits a fully faithful right adjoint by [HTT, 6.2.1.7].
Note also that, the bijection above specialises to

$$\mathrm{CovS}_\tau(x) = \{\text{covering sieves of } x\} \overset{1:1}{\leftrightarrow} \{\text{monomorphisms } U \hookrightarrow y(x) \mid L(U) \simeq Ly(x)\}.$$

In particular, this justifies the following Definition.

Definition 2.2.16. A localization $L\colon \mathcal{C} \to \mathcal{C}[S^{-1}]$ at a class of morphisms $S \subset \mathcal{C}_1$ is called **topological**, if the saturated closure $\bar{S}$ of S is generated by a class of monomorphisms in $\mathcal{C}$ and $\bar{S}$ is closed under pullbacks.

Examples 2.2.17. Let us end this subsection by considering a few familiar examples of ∞-topoi, obtained from pretopologies on 1-categories via Obs. 2.2.6.

1) **Open set topology:** Let Top denote the category of topological spaces. A covering for an $X \in \mathrm{Top}$ is a family $\{U_i \to X\}_{i \in I}$ of inclusions of open subsets $U_i \subset X$, such that $X = \cup_{i \in I} U_i$. If $\mathrm{Cov}_{\mathrm{open}}(X)$ denotes the collection of all coverings of X, then the assignment $X \mapsto \mathrm{Cov}_{\mathrm{open}}(X)$ specifies a pretopology τ_{open} on Top. The associated ∞-topos is denoted by $\mathrm{Shv}_{\mathrm{open}}(\mathrm{Top})$.

2) **Local Homeomorphisms:** One may also set coverings of an $X \in \mathrm{Top}$ to be the families of local homeomorphisms $\{X_i \to X\}_{i \in I}$ such that $\coprod_{i \in I} X_i \to X$ is surjective. If $\mathrm{Cov}_{\mathrm{lhomeo}}(X)$ denotes the collection of all such coverings of X, then the assignment $X \mapsto \mathrm{Cov}_{\mathrm{lhomeo}}$ specifies a pretopology τ_{lhomeo} on Top.

3) **Zariski topology:** The usual Zariski site $(\mathbf{Aff}, \tau_{\mathcal{Z}ar})$ is obtained by specifying a covering to be a family of morphisms $\{\iota_i^{op} : \mathrm{Spec}(A[\frac{1}{a_i}]) \to \mathrm{Spec}(A)\}_{i \in I}$ for $a_i \in A$ and ι_i being the localization, such that $\coprod_{i \in I} \iota_i^{op}$ is surjective. Denote the corresponding ∞-topos by $\mathrm{Shv}_{\mathrm{Zar}}(\mathrm{Aff})$.

4) **Étale topology:** The étale topology $\tau_{ét}$ on Aff is defined by setting the coverings of $A \in \mathrm{Aff}$ to be the families $\{\phi_i^{\mathrm{op}}\colon \mathrm{Spec}(A_i) \to \mathrm{Spec}(A)\}$, where each ϕ_i is an étale[4] ring homomorphism and $\coprod_{i \in I} \phi^{\mathrm{op}}$ is surjective. The corresponding ∞-topos is $\mathrm{Shv}_{ét}(\mathrm{Aff})$.

Definition 2.2.18. Let $(\mathcal{C}, \sigma)$ and $(\mathcal{D}, \tau)$ be ∞-presites. A functor $f\colon \mathcal{C} \to \mathcal{D}$ is a morphism of presites, if

- f is continuous i.e. for every $x \in \mathcal{C}$ and every covering $\{x_i \to x\}_{i \in I} \in \mathrm{Cov}_\sigma(x)$, the collection $\{f(x_i) \to f(x)\}_{i \in I}$ forms covering of $f(x)$ in τ;
- f preserves pullbacks along coverings i.e. for every $x \in \mathcal{C}$, any covering $\{x_i \to x\}_{i \in I} \in \mathrm{Cov}_\sigma(x)$ and any map $g\colon y \to x$, it holds that

$$f(y \times_x x_i) \simeq f(y) \times_{f(x)} f(x_i) \qquad \forall\, i \in I.$$

Example 2.2.19. Let $(\mathcal{C}, \tau)$ be a presite and $x \in \mathcal{C}$ an object. Then, $\mathcal{C}_{/x}$ inherits a pretopology from $\mathcal{C}$. If the context is non-ambiguous, one abbreviates $\mathrm{Shv}_\tau(x) := \mathrm{Shv}_\tau(\mathcal{C}_{/x})$. Hence, Ex. 2.2.17 provides the ∞-topoi $\mathrm{Shv}_{\mathrm{open}}(X)$ and $\mathrm{Shv}_{\mathrm{lhomeo}}(X)$ for $X \in \mathrm{Top}$, as well as $\mathrm{Shv}_{\mathrm{Zar}}(X)$ and $\mathrm{Shv}_{ét}(X)$ for $X \in \mathrm{Aff}$. Example 2.6.15 is going to verify, that these ∞-topoi coincide with the versions arising from the usual small sites such as $\mathrm{Op}(X)$.

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[4] See Appendix C.4 for a Definition and details of étale morphisms and the étale site.

2.3 Groupoids, Čech nerves and Effective Epimorphisms

Let $(\mathcal{C},\tau)$ be a site. Naturally one expects, that the objects in $\mathrm{Shv}_\tau(\mathcal{C})$ are characterised by a sheaf condition. The usual equalizer-condition (see [Har77]) will not do as it does not take in the higher coherencies, which constitute a homotopy equalizer. Equivalently, one would expect a morphisms of sheaves, which is induced by a covering, to behave like a quotient surjection of an equivalence relation i.e. an effective epimorphism.
Let us elaborate on this further. A relation on an object X of a finitely bicomplete[5] 1-category C is an object $R \in C$ together with a monomorphism $r\colon R \hookrightarrow X \times X$ and R is an equivalence relation, if for any $Z \in C$

$$\mathrm{Hom}_C(Z,R) \hookrightarrow \mathrm{Hom}_C(Z,X) \times \mathrm{Hom}_C(Z,X)$$

is an equivalence relation on $\mathrm{Hom}_C(Z,X)$ in the usual sense[6] (c.f. [Bor94a, 2.5.2]). Now, any map $f\colon X \to Y$ induces an equivalence relation $X \times_Y X \hookrightarrow X \times X$. Then, an equivalence relation R on X is said to be effective, if

$$X/R := \mathrm{coeq}(R \underset{\mathrm{pr}_2 \circ r}{\overset{\mathrm{pr}_1 \circ r}{\rightrightarrows}} X)$$

exists and $R \cong X \times_{X/R} X$ (see [Bor94a, 2.5.3]). Finally, an effective epimorphism is a map $X \to Y$ such that $Y \cong X/(X \times_Y X)$ and in ordinary topos theory all coverings produce effective epimorphism of sheaves in this sense.

For sets, the effective epimorphisms coincide with the usual surjections. For anima the situation is quite different. There, a map $X \to Y$ is a surjection, if $\pi_0 X \to \pi_0 Y$ is. However, surjections are not effective epimorphisms, as the map from the homotopy pullback $X \times_Y X \to X \times X$ does not even define a relation on X, because it is in general "not injective in any reasonable sense"[7] [HTT, p.431]. However, the elements of $X \times_Y X$ are paths (cf. [Mal25]), which naturally bear a coherently associative composition. Identifying all points that are connected via paths provides a quotient map, which is a surjective on connected components. This suggests to consider "relations up to coherent homotopy" and their "homotopy quotients", instead. For a general ∞-category $\mathcal{C}$ with finite limits, this amounts to extending the diagram $X \times_Y X \underset{r \circ \mathrm{pr}_2}{\overset{r \circ \mathrm{pr}_1}{\rightrightarrows}} X$ of the usual "relation" infinitely to the left so that it reads

$$\cdots \,\substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}\, X \times_Y X \times_Y X \times_Y X \,\substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}\, X \times_Y X \times_Y X \,\substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}\, X \times_Y X \underset{d_1}{\overset{d_0}{\rightrightarrows}} X \longrightarrow Y$$

This diagram $X^{\times_Y^\bullet} \to Y$ is an *augmented simplical* object, which are denoted as follows.

Notation 2.3.1. Write $\mathcal{C}^{\Delta^{\mathrm{op}}} := \mathrm{Fun}(\mathrm{N}(\Delta^{op}),\mathcal{C})$ for the ∞-category of **simplical objects** in $\mathcal{C}$ and denote by $\mathcal{C}^{\Delta_+^{\mathrm{op}}} := \mathrm{Fun}(\mathrm{N}(\Delta_+^{op}),\mathcal{C})$ the **augmented simplicial objects** in $\mathcal{C}$.

Thinking of an equivalence relation $x \sim y$ as an arrow $x \to y$, the simplicial object above must provide identity morphisms (by reflexivity) and a notion of composition (by transitivity),

---

[5] i.e. with all finite limits and finite colimits

[6] i.e. for $f,g \in \mathrm{Hom}_C(Z,X)$, one sets $f \sim g$ iff $(f,g) \in \mathrm{Hom}_C(Z,R) \subset \mathrm{Hom}_C(Z,X) \times \mathrm{Hom}_C(Z,X)$ and then $\sim$ should be an equivalence relation.

[7] Simply consider the extreme case $X = *$. Then, $X \times_Y X$ is the loop space ΩY based at the given point $X \to Y$. There is no reason for $\Omega Y \to *$ to be injective somehow.

in order to resemble an "homotopy equivalence relation". Consequently, the diagram needs to resemble category structure with objects X and arrows $X \times_Y X$. Moreover, the symmetry of $\sim$ imposes the condition that all morphisms arising from $\sim$ have to have inverses[8]. In sum, to model a "homotopy equivalence relation", $X^{\times_Y^\bullet}$ has to equip X with the data of a groupoid.

To formulate an appropriate condition for a category structure, first recall, that a monoid carries the exact same data as a 1-category with only one object. Conversely, a category resembles a many-object-version of a monoid, in the same way as groupoids do for groups. Following this perspective monoids and category structures internal to ∞-categories – so called Segal objects – are modelled follows.

Definition 2.3.2 ([nLa25e], cf.[Gro61, 4.1])**.** Let $\mathcal{C}$ be an ∞-category with finite products. A **Segal object** in $\mathcal{C}$ is a simplicial object $M\colon \mathrm{N}(\Delta^{\mathrm{op}}) \to \mathcal{C}$, when viewing M_0 as a collection of objects and M_1 as morphisms, then, for $n \geq 2$, M_n models the compositions of morphisms of length n. This is expressed by the condition, that for every $[n] \in \Delta$ and $1 \leq i \leq n$ the maps $\rho_i\colon [1] \to [n]$ in Δ defined by $0 \mapsto i-1$ and $1 \mapsto i$ induce equivalences

$$M_n \xrightarrow{\simeq} \underbrace{M_1 \times_{M_0} M_1 \times_{M_0} \ldots \times_{M_0} M_1}_{\text{n-times}} .$$

Moreover, a monoid in $\mathcal{C}$ is a Segal object $M\colon \mathrm{N}(\Delta^{\mathrm{op}}) \to \mathcal{C}$, where $M_0 \simeq *$.

In this perspective the edges M_1 of a monoid $M\colon \mathrm{N}(\Delta^{\mathrm{op}}) \to \mathcal{C}$ resemble its elements. Hence,

$$m\colon M_1 \times M_1 \simeq M_2 \xrightarrow{d_1} M_1, \qquad (m,n) \simeq \begin{tikzcd} & * & \\ * \arrow[ur, "mn"] \arrow[rr, "m"'] & & * \arrow[ul, "n"'] \end{tikzcd} \mapsto (* \xrightarrow{mn} *)$$

can be viewed as multiplication. Since $M_n \simeq M_1^{\times^n}$, this results in a natural transformation

$$m\colon M \times M \to M$$

which is a (coherently associative) multiplication and leads us to the following definition.

Definition 2.3.3 (cf.[GGN15, 1.1])**.** For an ∞-category $\mathcal{C}$ with finite products, a monoid M in $\mathcal{C}$ is a **group**, if there is an inversion $i\colon M \to M$ such that the composition

$$M \xrightarrow{\Delta} M \times M \xrightarrow{id \times i} M \times M \xrightarrow{m} M$$

is equivalent to $id\colon M \to M$. Here, Δ is the diagonal and m the multiplication.
Let $\mathrm{Grp}_{\mathbb{E}_1}(\mathcal{C}) \subset \mathcal{C}^{\Delta^{\mathrm{op}}}$ denote the full subcategory of group objects in $\mathcal{C}$.

Remark 2.3.4. There are also commutative analogues for the definitions above.
Let Fin_p be the category of finite sets with partially defined maps. A **commutative** monoid in an ∞-category $\mathcal{C}$ with finite products is defined as a functor $M\colon \mathrm{Fin}_p \to \mathcal{C}$, such that $M(\emptyset)$ is a terminal object and for every $I \in \mathrm{Fin}_p$ the collection of partially defined maps $I \supseteq \{i\} \to \{i\}$

---

[8] Let us spell this out: Suppose $x \sim x$ corresponds to $x \xrightarrow{id_x} x$. Suppose $\sim$ is symmetric and $x \sim y$, which we interpret as $x \to y$. Then, there has to be a morphism $y \to x$ corresponding to $y \sim x$ such that $x \to y \to x$ is equivalent to id_x, since $x \sim y \sim x$ reduces to $x \sim x$.

induces equivalences $M(I) \simeq \prod_{i\in I} M(i)$ (see [Wag23, §7.3] for details). Similar to Def, 2.3.3, a commutative monoid M in $\mathcal{C}$ is a group object, if the shear map $s\colon M \times M \xrightarrow{\mathrm{pr}_1\times m} M \times M$ is an equivalence (see [GGN15, 1.1]). Let $\mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{C}) \subset \mathrm{Fun}(\mathrm{Fin}_p, \mathcal{C})$ be the full subcategory of commutative group objects in $\mathcal{C}$. As for ordinary categories, there is a forgetful functor $\mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{C}) \to \mathrm{Grp}_{\mathbb{E}_1}(\mathcal{C})$, induced by the functor $\mathrm{Cut}\colon \Delta^{\mathrm{op}} \to \mathrm{Fin}_p$, which sends $[n]$ to the set of partitions of $[n]$ into non-empty intervals i.e. $\mathrm{Cut}([n]) \cong \{1, ..., n\}$. Then, morphisms $\alpha\colon [m] \to [n]$ automatically induce morphisms on the partitions (see [Wag23, 7.26] for details).

Note. $\mathbb{E}_1$ and $\mathbb{E}_\infty$ are used purely notational to indicate associativity and commutativity and have no further meaning in this thesis. To learn about the tower of commutativity, whose extreme cases are $\mathbb{E}_1$ and $\mathbb{E}_\infty$, the interested reader may consult [HA, §5].

Observation 2.3.5. Let $M \in \mathcal{C}^{\Delta^{\mathrm{op}}}$ be a monoid in a ∞-category $\mathcal{C}$ with finite products. The existence of an inversion map $i\colon M \to M$ is equivalent to requiring, that the shear map

$$s\colon M \times M \xrightarrow{pr_1\times m} M \times M$$

is an equivalence, since $id \times i$ must be the inverse to $pr_1 \times m$ up to contractible choice.
As noted in [GGN15, 1.1], both conditions are equivalent to requiring the monoid object in $h\mathcal{C}$ underlying M to be a group object, since $h(\text{-})$ is conservative and preserves products.

Example 2.3.6 (cf. [nLa25a, 2.3])**.** The reader might easily convinces herself, that a small 1-category C with objects C_0 and morphisms C_1 is a groupoid, if

$$s\colon C_1 \times_{C_0} C_1 \xrightarrow{pr_1\times m} C_1 \times_{C_0} C_1$$

is an isomorphism. For the transition to ∞-categories consider a Segal-object M in a 1-category C. Then, one may define the multiplication $m := d_1\colon M_2 \to M_1$, where M_2 is equivalent to $M_1 \times_{M_0} M_1$ by Def. 2.3.2. Hence, M_2 is the pullback of the square

$$\begin{tikzcd}
M_2 \arrow[r, "d_2"] \arrow[d, "d_0"] & M_1 \arrow[d, "d_0"] \\
M_1 \arrow[r, "d_1"] & M_0
\end{tikzcd}$$

since $\rho_0 = d_2$ and $\rho_1 = d_0$. Also, consider the square

$$\begin{tikzcd}
M_2 \arrow[r, "d_0"] \arrow[d, "m=d_1"'] & M_1 \arrow[d, "d_0"] \\
M_1 \arrow[r, "d_0"] & M_0.
\end{tikzcd}$$

Denoting the pullback of this square by $M_1 \times'_{M_0} M_1$, one obtains the “shear map”

$$s\colon M_1 \times_{M_0} M_1 \simeq M_2 \xrightarrow{\mathrm{pr}_1\times m := d_2\times d_1} M_1 \times'_{M_0} M_1$$

Hence, s is an equivalence if and only if M_2 is the pullback of both squares and one may define groupoid objects requiring this condition.
Taking all higher coherencies into account, one defines a ∞-groupoids as follows.

Definition 2.3.7. A simplicial object $U_\bullet \in \mathcal{C}^{\Delta^{\mathrm{op}}}$ is a groupoid object in $\mathcal{C}$, if for every $[n] \in \Delta$

and for any two subsets $S, S' \subset [n]$ with $S \cup S'$ and $S \cap S' = \{s\}$, the commutative diagram

$$\begin{tikzcd}
U([n]) \arrow[r] \arrow[d] & U(S) \arrow[d] \\
U(S') \arrow[r] & U(\{s\})
\end{tikzcd}$$

which is induced by the face maps, is a pullback in $\mathcal{C}$.

Then, Ex. 2.3.6 is recovered by unravelling[9] Def. 2.3.7 for the case n=2. By Obs. 2.3.5, a group object in $\mathcal{C}$ is a groupoid object $U\colon \mathrm{N}(\Delta^{\mathrm{op}}) \to \mathcal{C}$ where U_0 is a final object.

Example 2.3.8. Let $f\colon X \to Y$ be a map in an ∞-category $\mathcal{C}$ with pullbacks. Then, the simplicial object

$$X^{\times_Y^{\bullet}}\colon \mathrm{N}(\Delta^{\mathrm{op}}) \to \mathcal{C}, \quad [0] \mapsto X, \quad [n] \mapsto \underbrace{X \times_Y \cdots \times_Y X}_{\text{n-times}}.$$

is a groupoid object in $\mathcal{C}$.

Finally, we have all tools together to characterise the augmented simplicial objects, which resemble quotient surjections i.e. effective epimorphisms. To do so, we start by formalizing of augmented simplicial object of the form $X^{\times_Y^{\bullet}}$ by defining Čech nerves.

Notation 2.3.9. Let $\Delta^{\leq n} \subset \Delta$ and $\Delta_+^{\leq n} \subset \Delta_+$ be the full subcategories spanned by $[k]$ for $0 \leq k \leq n$ and $-1 \leq k \leq n$, respectively.

Definition 2.3.10. Let $\mathcal{C}$ be an ∞-category with pullbacks. An augmented simplicial object $U_\bullet^+ \in \mathcal{C}^{\Delta_+}$ is a **Čech nerve** if

- The underlying simplicial object of $U_\bullet^+$ is a groupoid object.
- The square $U_\bullet^+|_{\mathrm{N}(\Delta_+^{\leq 1})^{op}}$ i.e.

$$\begin{tikzcd}
U_1 \arrow[r] \arrow[d] & U_0 \arrow[d] \\
U_0 \arrow[r] & U_{-1}
\end{tikzcd}$$

  is a pullback square.

By the next Proposition, which is stated without proof, Čech nerves are Kan extensions.

Proposition 2.3.11 ([HTT, 6.1.2.11])**.** *Let $U_\bullet^+ \in \mathcal{C}^{\Delta_+}$ be an augmented simplicial object in a ∞-category. Then, the following statements are equivalent*

- *$U_\bullet^+$ is a Čech nerve.*
- *$U_\bullet^+$ is the right Kan extension of $U_\bullet^+|_{\mathrm{N}(\Delta_+^{\leq 0})}$ i.e. of $U_0 \to U_{-1}$.*

Hence, a Čech nerve is uniquely determined by its augmentation map $U_0 \to U_{-1}$.

Using Ex. 2.3.8, one immediately deduces the following Corollary.

---

[9] The first square in Ex. 2.3.6 corresponds to $[2] = \{0,1\} \cup_{\{1\}} \{1,2\}$ and the second square corresponds to $[2] = \{0,1\} \cup_{\{0\}} \{0,2\}$.

Corollary 2.3.12. *For any morphism $f : X \to Y$ in an ∞-category $\mathcal{C}$, which admits all pullbacks, the groupoid $\check{\mathrm{C}}(f) := X^{\times^\bullet_Y}$ of Ex. 2.3.8 is a Čech nerve with augmentation map is f. Likewise, if $U^+_\bullet$ is a Čech nerve with augmentation map $u : U^+_0 \to U^+_{-1}$, then $U^+_\bullet \simeq \check{\mathrm{C}}(u)$.*

As outlined above, the equivalence relations in an ∞-category $\mathcal{C}$ are precisely the groupoid objects of $\mathcal{C}$. Furhtermore, the condition for an equivalence relation to be effective coincides with the condition of a groupoid to be a Čech nerve. This justifies the following Definition.

Definition 2.3.13. Let $\mathcal{C}$ a be an ∞-category with pullbacks. A groupoid $U_\bullet \in \mathcal{G}rpd(\mathcal{C})$ is **effective**, if $U_\bullet$ can be extended to a colimit diagram $U^+_\bullet : \mathrm{N}(\Delta^+) \to \mathcal{C}$, such that $U^+_\bullet$ is a Čech nerve.
A morphism $f\colon X \to Y$ in $\mathcal{C}$ is an **effective epimorphism** if $\check{C}(f)$ is a colimit diagram.

Observation 2.3.14. All morphisms in $\mathcal{LT}$op preserve effective epimorphisms, since they preserve small colimits and finite limits.

One of the central results of Lurie [HTT], building on ideas of Joyal, Toen-Vezzosi and Rezk, was to give a intrinsic Characterisation of ∞-topoi parallel to Giraud's Characterisation of 1-topoi (see [Bor94b, 3.6.1]). The main insight for the transition from 1-topoi to ∞-topoi is that effective groupoids resemble effective equivalence relations, as outlined above. All other conditions of [Bor94b, 3.6.1] can be transferred directly to ∞-categories.

Theorem 2.3.15 (∞-Giraud axioms, [HTT, 6.1.0.6])**.** *A presentable ∞-category $\mathcal{X}$ is an ∞-topos if and only if*

1) *Coproducts in $\mathcal{X}$ are disjoint i.e. for every $X, Y \in \mathcal{X}$ the canonical maps $X \to X \coprod Y$ and $Y \to X \coprod Y$ are monomorphisms and $X \times_{X \coprod Y} Y$ is an initial object.*

2) *Colimits are universal i.e. colimits are stable under pullback.*[10]

3) *All groupoids in $\mathcal{X}$ are effective.*

These conditions exhibit ∞-topoi as necessary and sufficient environments for homotopical constructions like homotopy groups or Eilenberg-MacLane objects, which will be discussed in the next sections.

By the next proposition, which is stated without proof, effective epimorphism act as counterpart to monomorphisms in the same way as usual epi- and monomorphisms do.

Proposition 2.3.16 ([HTT, 5.2.8.16, 7.2.1.14])**.** *The classes of monomorphisms and effective epimorphisms in an ∞-topos $\mathcal{X}$ form a "orthogonal factorization system". In particular, every morphism $f\colon X \to Y$ in $\mathcal{X}$ is equivalent to a unique composition*

$$X \overset{g}{\twoheadrightarrow} Z \overset{h}{\hookrightarrow} Y$$

where Z is some object, g is an effective epimorphism and h is a monomorphism in $\mathcal{X}$. Such factorization is called an ***epi/mono*** *factorization.*

---

[10] See Def. A.2.4 for equivalent formulations.

Corollary 2.3.17 ([HTT, 6.2.3.16],[HTT, 6.2.3.15]). *Let $\mathcal{X}$ be an ∞-topos and let*

$$\begin{tikzcd}
X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
S' \arrow[r, "g"'] & S
\end{tikzcd}$$

be a pullback square in $\mathcal{X}$.

1) If g is an effective epimorphism, then g' is an effective epimorphism.

2) If f' is an equivalence and g is an effective epimorphism, then f is an equivalence, as well.

Prop. 2.3.16 also enables us to show that coverings induce effective epimorphisms.

Lemma 2.3.18. *Let $(\mathcal{C}, \tau)$ be a presite and $\{x_i \xrightarrow{f_i} x\}_{i \in I}$ a covering in $\mathcal{C}$. Then,*

$$\coprod_{i \in I} Ly(f_i) \colon \quad \coprod_{i \in I} Ly(x_i) \to Ly(x)$$

is an effective epimorphism in $\mathrm{Shv}_\tau(\mathcal{C})$, where L is the localization $L \colon \mathcal{P}(\mathcal{C}) \to \mathrm{Shv}_\tau(\mathcal{C})$.

Proof. Let $S_x \subset \mathcal{C}_{/x}$ be the covering sieve generated by $\{x_i \to x\}_{i \in I}$ and let $\coprod_i y(x_i) \twoheadrightarrow V \to y(x)$ be a epi/mono factorization of $\coprod_i y(f_i)$. As explained in Rmk. 2.2.15, there is an unique monomorphism $U \hookrightarrow y(x)$, such that for every $c \to x$ in S_x, the induced morphism $y(c) \to y(x)$ factors through U However, a morphism $c \to x$ of $\mathcal{C}$ is in S_x precisely if it factors through an $x_i \to x$. Therefore, $V \hookrightarrow y(x)$ must be the monomorphism given by the correspondence of Rmk. 2.2.15, i.e. $U \simeq V$. Moreover, $L(U) \simeq L(y(x))$, since L is precisely the localization at such monomorphisms $U \hookrightarrow y(x)$ by Rmk. 2.2.15. Then,

$$\coprod_i Ly(x_i) \twoheadrightarrow L(U) \simeq Ly(x)$$

is an effective epimorphism, as L preserves effective epimorphisms by Obs. 2.3.14. ☐

Finally, the newly introduced notion of effective epimorphisms enables us to phrase a higher categorical analogue to the usual sheaf condition.

Proposition 2.3.19 ([AHW17, 3.13]). *Let $(\mathcal{C}, \tau)$ be a site, $x \in \mathcal{C}$ and $\{u_i \to x\}_{i \in I}$ be a family of morphisms generating*[11] *a covering sieve $S_x \in \mathrm{CovS}_\tau(x)$.*
Let $\check{C}_\bullet := \check{C}(\coprod_{i \in I} y(u_i) \to y(x))_\bullet$ be the associated Čech nerve in $\mathcal{P}(\mathcal{C})$.
Then, a presheaf F is S_x-local [12] *if and only if the induced map*

$$F(x) \to \lim_{n \in \mathrm{N}(\Delta^{\mathrm{op}})} \mathrm{map}_{\mathcal{P}(\mathcal{C})}(\check{C}_n, F)$$

is an equivalence. In particular, if the Čech nerve $\check{c}_\bullet := \check{C}(\coprod_{i \in I} u_i \to x)_\bullet$ exists in $\mathcal{C}$[13]*, the condition above simplifies to*

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[11] This is the case, for example if $(\mathcal{C}, \tau)$ is the site associated to a presite and $\{u_i \to x\}_{i \in I}$ is a covering.

[12] See Def. 2.2.10

[13] For example, if $\mathcal{C}$ admits all pullbacks and coproducts.

$$F(x) \quad \simeq \quad \lim_{n\in \mathrm{N}(\Delta^{\mathrm{op}})} F(\check{c}_n) \quad \simeq \quad \lim\big(\prod_{i\in I} F(u_i) \rightrightarrows \prod_{i,j\in I} F(u_i \times_x u_j) \overset{\rightarrow}{\rightrightarrows} \cdots\big)$$

Proof. By the proof of Lem. 2.3.18, the composition

$$\coprod_{i\in i} y(u_i) \twoheadrightarrow \mathrm{colim}_{[n]\in \mathrm{N}(\Delta^{\mathrm{op}})} \check{C}_n \hookrightarrow y(x)$$

is an epi-mono factorization in $\mathcal{P}(\mathcal{C})$ and $\mathrm{colim}_{[n]\in \mathrm{N}(\Delta^{\mathrm{op}})} \check{C}_n \hookrightarrow y(x)$ is equivalent to the monomorphism corresponding to S_x introduced in Rmk. 2.2.15. Therefore,

$$F \text{ is } S_x\text{-local} \overset{2.2.15}{\iff} F(x) \simeq \mathrm{map}_{\mathcal{P}(\mathcal{C})}\big(\operatorname*{colim}_{[n]\in \mathrm{N}(\Delta^{\mathrm{op}})} \check{C}_n, F\big) \iff F(x) \simeq \lim_{[n]\in \mathrm{N}(\Delta^{\mathrm{op}})} \mathrm{map}_{\mathcal{P}(\mathcal{C})}(\check{C}_n, F).$$

If $\check{c}_\bullet$ exists, then $\check{C}_\bullet = y \circ \check{c}_\bullet$ and thus,

$$F \text{ is } S_x\text{-local} \iff F(x) \simeq \lim_{[n]\in \mathrm{N}(\Delta^{\mathrm{op}})} F(\check{c}_n)$$

and $F(\check{c}_\bullet)\colon \mathrm{N}(\Delta^{\mathrm{op}}) \to \mathrm{Ani}$ is the diagram appearing in the simplified condition. □

It remains to treat the question, when a functor from a (pre)site $\mathcal{C}$ to an ∞-topos $\mathcal{X}$ induces a geometric morphism $\mathrm{Shv}(\mathcal{C}) \to \mathcal{X}$. As as preliminary step, the next example explicates, how a functor $\mathcal{C} \to \mathcal{D}$ of ∞-categories induces an adjunction between the respective presheaves via Kan extension.

Example 2.3.20. Let $f\colon \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. By [Lan21, 4.3.37] the ∞-category Cat_∞ is bicomplete. Thus, all slice categories arising from Cat_∞ are complete by [Lan21, 4.3.35]. Thus, pulling back along f gives a functor[14]

$$f^*\colon \mathrm{Cat}_{\infty/\mathcal{D}} \to \mathrm{Cat}_{\infty/\mathcal{C}},\ (\mathcal{E} \to \mathcal{D}) \mapsto (\mathcal{E} \times_{\mathcal{D}} \mathcal{C} \to \mathcal{C}).$$

Since right fibrations are stable under pullback, this functor restricts to

$$f^*\colon \ \mathrm{RFib}(\mathcal{D}) \to \mathrm{RFib}(\mathcal{C})$$

where RFib denotes the full subcategory of right fibrations. Straightening[15] f^* yields

$$f^*\colon \mathcal{P}(\mathcal{D}) \to \mathcal{P}(\mathcal{C}),\ (F\colon d \mapsto E_d) \mapsto (f^*F\colon c \mapsto (E \times_{\mathcal{D}} \mathcal{C})_c \simeq E_{f(c)}).$$

Since $\mathcal{P}(\mathcal{D})$ is cocomplete this functor admits a left adjoint $f_!$, which is the Kan extension of $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{y} \mathcal{P}(\mathcal{D})$ along the Yoneda embedding and is calculated pointwise as[16]

$$f_!(P) := \mathrm{colim}\,(\mathcal{C}_{/P} \to \mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{y} \mathcal{P}(\mathcal{D})) = \operatorname*{colim}_{(x,\, y(x)\to P(x))\in \mathcal{C}_{/P}} y(f(x))$$

---

[14] This functor is explicitly constructed in Cor. A.1.5

[15] cf. [Lan21, §3.3]

[16] See [Wag23, 6.30] for details.

for every $P \in \mathcal{P}(\mathcal{C})$. Hence, setting $g_* := f^*$ and $g^* := f_!$ provides a geometric morphism

$$\mathcal{P}(\mathcal{D}) \underset{g_*}{\overset{g^*}{\leftrightarrows}} \mathcal{P}(\mathcal{C}).$$

Suppose $f\colon \mathcal{C} \to \mathcal{D}$ is not merely a functor, but a morphism of presites and consider

$$F\colon \mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{y} \mathcal{P}(\mathcal{D}) \xrightarrow{L} \mathrm{Shv}(\mathcal{D}).$$

For the construction of a geometric morphism $\mathrm{Shv}(\mathcal{C}) \to \mathrm{Shv}(\mathcal{D})$ extending F, it is essential to determine when the left Kan extension $F_!\colon \mathcal{P}(\mathcal{C}) \to \mathrm{Shv}(\mathcal{D})$ of the functor F descends to a functor $\tilde{F}_!\colon \mathrm{Shv}(\mathcal{C}) \to \mathrm{Shv}(\mathcal{D})$. The next Lemma provides an answer to this question.

Lemma 2.3.21 ([HTT, 6.2.3.19]). *Let $\mathcal{X}$ be an ∞-topos, $(\mathcal{C}, \tau)$ a small presite and let $\mathcal{X} \underset{f_*}{\overset{f^*}{\leftrightarrows}} \mathcal{P}(\mathcal{C})$ be a geometric morphism. Then, the following are equivalent*

1) *f_* factors through $\mathrm{Shv}_\tau(\mathcal{C}) \subset \mathcal{P}(\mathcal{C})$.*
2) *For every covering $\{x_i \to x\}_{i \in I}$ in $\mathcal{C}$, the induced map $\coprod_i f^*(y(x_i)) \longrightarrow f^*(y(x))$ is an effective epimorphism in $\mathcal{X}$.*

Proof. 1) ⇒ 2): Let $L\colon \mathcal{P}(\mathcal{C}) \to \mathrm{Shv}_\tau(\mathcal{C})$ be the canonical localization. By assumption, there is a geometric morphism $g^*\colon \mathrm{Shv}_\tau(\mathcal{C}) \to \mathcal{X}$ such that $f^* \simeq g^* \circ L$. The assertion already follows, since $\coprod_i Ly(x_i) \to Ly(x)$ is a effective epimorphism by Lem. 2.3.18 and g^* preserves geometric epimorphisms.

2) ⇒ 1): Recall from Rmk. 2.2.15 that $\mathrm{Shv}_\tau(\mathcal{C})$ is the localization of $\mathcal{P}(\mathcal{C})$ at the class of monomorphisms $U \to y(x)$, which correspond to sieves generated by coverings. Let $m\colon U \hookrightarrow y(x)$ be such a monomorphism, which is associated to a covering $\{x_i \to x\}$. As such, U is the intermediate object of a epi/mono factorization as in the proof of Lem. 2.3.18

$$h\colon \coprod_i y(x_i) \overset{e}{\twoheadrightarrow} U \overset{m}{\hookrightarrow} y(x).$$

Hence, $f^*(e)$ is an effective epimorphism by Obs. 2.3.14. By assumption $f^*(h)$ is an effective epimorphism and therefore Lem. 2.3.17 implies, that m is an equivalence. □

In case $\mathcal{P}(\mathcal{C}) \to \mathcal{X}$ arises from a morphism of presites, Lem. 2.3.21 spells out to the following fundamental result.

Corollary 2.3.22. *Any morphism of presites $f\colon (\mathcal{C}, \sigma) \to (\mathcal{D}, \tau)$ gives rise to a geometric morphism $\mathrm{Shv}_\tau(\mathcal{D}) \underset{g_*}{\overset{g^*}{\leftrightarrows}} \mathrm{Shv}_\sigma(\mathcal{C})$, where g_* is the precomposition with f as in Ex. 2.3.20.*

Proof. By assumption, every covering $\{x_i \to x\}_{i \in I} \in \mathrm{Cov}_\sigma(x)$ induces a covering $\{f(x_i) \to f(x)\} \in \mathrm{Cov}_\tau(f(x))$ for any $x \in \mathcal{C}$. Since,

$$\coprod_i f_! y(x_i) \to f_! y(x) \qquad \simeq \qquad \coprod_i L_\tau y(f(x_i)) \to L_\tau y(f(x))$$

and the right side is an effective epimorphism by Lem. 2.3.18, the precomposition functor $\mathrm{Shv}_\tau(\mathcal{D}) \subset \mathcal{P}(\mathcal{D}) \xrightarrow{f^*} \mathcal{P}(\mathcal{C})$ actually lands in $\mathrm{Shv}_\sigma(\mathcal{C})$ by Lem. 2.3.21. ☐

As another consequence of Lem. 2.3.21, all geometric morphisms from $\mathrm{Shv}_\tau(\mathcal{C})$ arise from functors, which send coverings to effective epimorphisms. This is phrased as follows.

Theorem 2.3.23 ([HTT, 6.2.3.20])**.** *Let $\mathcal{X}$ be an ∞-topos, let $(\mathcal{C}, \tau)$ be a small presite and let $L : \mathcal{P}(\mathcal{C}) \to \mathrm{Shv}_\tau(\mathcal{C})$ be the canonical localization. Then, the functor*

$$J : \mathrm{Fun}^*(\mathrm{Shv}(\mathcal{C}), \mathcal{X}) \xrightarrow{L^*} \mathrm{Fun}^*(\mathcal{P}(\mathcal{C}), \mathcal{X}) \xrightarrow{y^*} \mathrm{Fun}(\mathcal{C}, \mathcal{X})$$

obtained by precomposing with L and y, is fully faithful.

Suppose further that $\mathcal{C}$ admits finite limits. Then, a functor $f : \mathcal{C} \to \mathcal{X}$ lies in the essential image of J if and only if:

1) *f is left exact.*
2) *For every covering family $\{x_i \to x\}_{i \in I}$ in $\mathcal{C}$, the induced map $\coprod_{i\in I} f(x_i) \to f(x)$ is an effective epimorphism in $\mathcal{X}$.*

Proof. By [Lan21, 2.4.2] and Rmk. 2.2.15, precomposition with L induces a fully faithful functor $\mathrm{Fun}(\mathrm{Shv}_\tau(\mathcal{C}), \mathcal{X}) \to \mathrm{Fun}(\mathcal{P}(\mathcal{C}), \mathcal{X})$, which restricts to L^*, since L preserves finite limits and colimits. Likewise, precomposing with y yields a fully faithful functor $\mathrm{Fun}(\mathcal{P}(\mathcal{C}), \mathcal{X}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{X})$, which restricts to y^*. Hence, J is fully faithful.
If $\mathcal{C}$ admits finite limits and $f \colon \mathcal{C} \to \mathcal{X}$ is a left exact functor, then there is a geometric morphism $g^* \colon \mathcal{P}(\mathcal{C}) \to \mathcal{X}$, such that $f \simeq y^*(g^*)$. Then, by Lem. 2.3.21, g^* belongs to the essential image of L^* if and only if f satisfies the condition 2) above. ☐

An application of Thm. 2.3.23 yields that anima occupy an universal role among ∞-topoi.

Corollary 2.3.24. Ani *is an initial object in* $\mathcal{LT}$op *and thus terminal in* $\mathcal{RT}$op.

Proof. Let $\mathcal{X}$ be a ∞-topos. Since $\mathrm{An} = \mathcal{P}(\Delta^0)$, Thm. 2.3.23 implies that

$$\mathrm{Fun}^*(\mathrm{Ani}, \mathcal{X}) \simeq \mathrm{Fun}^*(\Delta^0, \mathcal{X}) \simeq \mathcal{X}_{\mathrm{term}} \simeq *$$

where $\mathcal{X}_{\mathrm{term}} \subset \mathcal{X}$ is the full subcategory of terminal objects, which is contractible by [Lan21, Prop.4.1.3], since $\mathcal{X}$ is complete. Therefore, Ani is an initial object in $\mathcal{LT}$op. ☐

The next Lemma provides an explicit description of the geometric morphism $\mathrm{Ani} \to \mathcal{X}$.

Lemma 2.3.25. *Let $\mathcal{X}$ be a ∞-topos and $\mathcal{X} \underset{i}{\overset{L}{\leftrightarrows}} \mathcal{P}(\mathcal{C})$ a localization for a small ∞-category $\mathcal{C}$. Denote the constant presheaf functor by*

$$\underline{\Delta}_{\mathcal{P}(\mathcal{C})} \colon \mathrm{Ani} \to \mathcal{P}(\mathcal{C}),\ K \mapsto (\mathcal{C}^{\mathrm{op}} \to \Delta^0 \xrightarrow{K} \mathrm{Ani})$$

Pick a $1_{\mathcal{P}(\mathcal{C})} \in \mathcal{P}(\mathcal{C})_{\mathrm{term}}$*. Then,* $\underline{\Delta}_{\mathcal{P}(\mathcal{C})}$ *is equivalent to the tensor product*[17] *with anima*

$$- \cdot 1_{\mathcal{P}(\mathcal{C})} \colon \mathrm{Ani} \to \mathcal{P}(\mathcal{C}).$$

Moreover, setting $\Gamma_{\mathcal{X}} := \mathrm{map}_{\mathcal{X}}(1_{\mathcal{X}}, -)$ *and* $\underline{\Delta}_{\mathcal{X}} := L \circ \underline{\Delta}_{\mathcal{P}(\mathcal{C})}$ *gives rise to the geometric morphism*

$$\mathcal{X} \underset{\underline{\Delta}_{\mathcal{X}}}{\overset{\Gamma_{\mathcal{X}}}{\rightleftarrows}} \mathrm{Ani}.$$

Proof. Let $\tilde{1} \colon \Delta^0 \to \mathcal{P}(\mathcal{C})$ be the functor with target $1_{\mathcal{P}(\mathcal{C})}$. By Cor. 2.3.24 and Ex. 2.3.20, the essentially unique morphism $\mathrm{Ani} \to \mathcal{P}(\mathcal{C})$ is the left Kan extension of $\tilde{1}$, given by

$$\tilde{1}_!(K) = \operatorname{colim}\left(\Delta^0_{/K} \to \Delta^0 \xrightarrow{\tilde{1}} \mathcal{P}(\mathcal{C})\right) \simeq \operatorname{colim}\left(K \to \Delta^0 \xrightarrow{\tilde{1}} \mathcal{P}(\mathcal{C})\right) \overset{A.3.2}{\simeq} K \cdot 1_{\mathcal{P}(\mathcal{C})}.$$

Since (co)limits in $\mathcal{P}(\mathcal{C})$ are calculated pointwise, $1_{\mathcal{P}(\mathcal{C})}$ is the functor $\underline{\Delta}(*) \colon \mathcal{C}^{\mathrm{op}} \to \Delta^0 \xrightarrow{*} \mathrm{Ani}$ and therefore $K \cdot 1_{\mathcal{P}(\mathcal{C})}$ is the functor sending every $c \in \mathcal{C}$ to $\operatorname{colim}_K * \simeq K \in \mathrm{Ani}$. Thus, $K \cdot 1_{\mathcal{P}(\mathcal{C})} \simeq \underline{\Delta}_{\mathcal{P}(\mathcal{C})}(K)$ for all $K \in \mathrm{Ani}$. By the equivalences[18] above, the tensor product $\text{-} \cdot 1_{\mathcal{P}(\mathcal{C})}$ is a pointwise a colimit and sends $*$ to $1_{\mathcal{P}(\mathcal{C})}$. Hence, $\text{-} \cdot 1_{\mathcal{P}(\mathcal{C})}$ preserves colimits and finite limits, since colimits are universal in $\mathcal{P}(\mathcal{C})$.

To show, that $\Gamma_{\mathcal{X}}$ and $\underline{\Delta}_{\mathcal{X}}$ form an adjunction, recall that for any $X \in \mathcal{X}$, there is a $P \in \mathcal{P}(\mathcal{C})$ such that $L(P) \simeq X$, as L is essentially surjective. Let i denote the inclusion $\mathrm{Shv}_\tau(\mathcal{C}) \subset \mathcal{P}(\mathcal{C})$. Then, there is a series of equivalences

$$\begin{aligned} &\mathrm{map}_{\mathcal{X}}(\underline{\Delta}_{\mathcal{X}}(K), X) \simeq \mathrm{map}_{\mathcal{X}}(L\,\underline{\Delta}_{\mathcal{P}(\mathcal{C})}(K), L(P)) \simeq \mathrm{map}_{\mathcal{P}(\mathcal{C})}(\underline{\Delta}_{\mathcal{P}(\mathcal{C})}(K), iLP) \\ \simeq\, &\mathrm{map}_{\mathcal{P}(\mathcal{C})}(K \cdot 1_{\mathcal{P}(\mathcal{C})}, iLP) \simeq \mathrm{map}_{\mathrm{Ani}}(K, \mathrm{map}_{\mathcal{P}(\mathcal{C})}(1_{\mathcal{P}(\mathcal{C})}, iLP)) \simeq \mathrm{map}_{\mathrm{Ani}}(K, \mathrm{map}_{\mathcal{X}}(1_{\mathcal{X}}, LP)) \\ &\simeq \mathrm{map}_{\mathrm{Ani}}(K, \mathrm{map}_{\mathcal{X}}(1_{\mathcal{X}}, X)) = \mathrm{map}_{\mathrm{Ani}}(K, \Gamma_{\mathcal{X}}) \end{aligned}$$

□

Definition 2.3.26. $\underline{\Delta}_{\mathcal{X}}$ is called the **constant sheaf functor** and $\Gamma_{\mathcal{X}}$ is called the **global sections functor** of $\mathcal{X}$. If unambiguous, one denotes $\underline{K} := \underline{\Delta}_{\mathcal{X}}(K)$ for any $K \in \mathrm{Ani}$.

The global section functor $\Gamma_{\mathcal{X}}$ resembles the usual global sections Γ in the following way.

Example 2.3.27. Let X be a topological space. Then, $Ly(X) \in \mathrm{Shv}_{\mathrm{open}}(X)$ is the terminal object, because X is the terminal object of $\mathrm{Top}_{/X}$ and y and L preserve terminal objects. Then, the Yoneda Lemma yields for every $F \in \mathrm{Shv}_{\mathrm{open}}(X)$:

$$\Gamma_{\mathrm{Shv}_{\mathrm{open}}(X)}(F) = \mathrm{map}_{\mathrm{Shv}_{\mathrm{open}}(X)}(Ly(X), F) \simeq \mathrm{map}_{\mathcal{P}(X)}(y(X), iF) \simeq F(X).$$

Before coming to § 2.4, where we show that slice-categories formed from ∞-topoi classify sections, let us note an application of Lem. 2.3.25.

Recall that every ∞-topos admits tensors $K \cdot \text{-}$ and cotensors -^K over anima, such as internal mapping spaces $\underline{\mathrm{Hom}}$, all defined in § A.3. For ∞-topoi these notions relate as follows.

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[17] See Def. A.3.1

[18] See also Lem. A.3.2

Lemma 2.3.28. *For every ∞-topos $\mathcal{X}$, there are equivalences*

$$1.\ K \cdot X \simeq \underline{K} \times X \qquad\qquad 2.\ Y^K \simeq \underline{\mathrm{Hom}}_{\mathcal{X}}(\underline{K}, Y)$$

which are natural in $K \in \mathrm{Ani}$, $X, Y \in \mathcal{X}$.

Proof. Since $\underline{\Delta}$ and $- \times X \colon \mathcal{X} \to \mathcal{X}$ preserve small colimits, we have that

$$\underline{K} \times X \cong \underline{\Delta}(\operatorname*{colim}_K *) \times X \simeq \operatorname*{colim}_K \underline{*} \times X \simeq \operatorname*{colim}_K 1_{\mathcal{X}} \times X \simeq K \cdot X.$$

The second claim is verified by applying the Yoneda Lemma to the natural equivalence

$$\mathrm{map}_{\mathcal{X}}(X, Y^K) \simeq \mathrm{map}_{\mathcal{X}}(K \cdot X, Y) \simeq \mathrm{map}_{\mathcal{X}}(\underline{K} \times X, Y) \simeq \mathrm{map}_{\mathcal{X}}(X, \underline{\mathrm{Hom}}_{\mathcal{X}}(\underline{K}, Y)). \qquad \square$$

Thus, every f^* in an essential geometric morphism commutes with $\underline{\mathrm{Hom}}$. More generally:

Corollary 2.3.29. *Let $f^* \colon \mathcal{X} \to \mathcal{Y}$ be a limit-preserving morphism between ∞-topoi. Then, for every $X \in \mathcal{X}$ and $K \in \mathrm{Ani}$,*

$$f^*(\underline{\mathrm{Hom}}_{\mathcal{X}}(\underline{K}, X)) \simeq f^*(X^K) \overset{A.3.3}{\simeq} f^*(\lim_K X) \simeq \lim_K f^*(X) \simeq f^*(X)^K \simeq \underline{\mathrm{Hom}}_{\mathcal{Y}}(\underline{K}, f^*(X)).$$

In particular, if $f^* \colon \mathcal{X} \to \mathcal{Y}$ is a limit-preserving morphism in $\mathcal{LT}\mathrm{op}$, then

$$f^*(\underline{\mathrm{Hom}}_{\mathcal{X}}(\underline{K}, X)) \simeq \underline{\mathrm{Hom}}_{\mathcal{Y}}(f^*(\underline{K}), f^*(X)).$$

2.4 Slices of ∞-Topoi

For any morphism $f \colon X \to Y$ presentable ∞-category with universal colimits, the standard adjunction[19] $f_! \dashv f^*$ between slice categories extends to an adjoint triple $f_! \dashv f^* \dashv f_*$, as f^* preserves colimits (cf. [Lan21, 5.2.4]). A key step towards constructing a classifying topos for Γ is to observe that this triple forms an essential geometric morphism in the context of ∞-topoi.

Definition 2.4.1 ([Joh02, A.4.1.5])**.** An **essential geometric morphism** between ∞-topoi $\mathcal{X}$ and $\mathcal{Y}$ is a geometric morphism $\mathcal{X} \overset{f^*}{\underset{f_*}{\leftrightarrows}} \mathcal{Y}$, where f^* admits a further left adjoint $f_!$. In this case, the situation is displayed as

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "f_!"] & \mathcal{Y} \arrow[l, "f^*" description] \arrow[from=1-1, to=1-2, bend right, "f_*"'] \end{tikzcd}.$$

Let $\mathcal{RT}\mathrm{op}^{\mathrm{ess}} \subset \mathcal{RT}\mathrm{op}$ and $\mathcal{LT}\mathrm{op}^{\mathrm{ess}} \subset \mathcal{LT}\mathrm{op}$ denote the subcategories[20] containing only essential geometric morphisms.

Warning 2.4.2. The functor $f_! \colon \mathcal{Y} \to \mathcal{X}$ participating in an essential geometric morphism is **not left exact** in general. Therefore $\mathcal{Y} \overset{f_!}{\underset{f^*}{\leftrightarrows}} \mathcal{X}$ is not a geometric morphism.

---

[19] See Cor. A.1.5.

[20] By the usual properties of adjunctions, essential geometric morphisms are closed under composition and contain all equivalences.

Lemma 2.4.3. *Let $\mathcal{X}$ be an ∞-topos and $U \in \mathcal{X}$. Then $\mathcal{X}_{/U}$ is an ∞-topos.*

The proof of Lem. 2.4.3, which is similar to [HTT, 6.3.5.1], but slightly more streamlined, requires the following characterisation of ∞-topoi, which is stated without proof.

Theorem 2.4.4 (Rezk, [HTT, 6.1.0.6, 6.1.3.9])**.** *For every presentable ∞-category $\mathcal{C}$ the following conditions are equivalent*

1) *$\mathcal{C}$ is an ∞-topos.*

2) *Let $\bar\alpha\colon \bar p \to \bar q$ be a natural transformation of cocones $\bar p, \bar q\colon K\star\Delta^0\colon \mathcal{C}$ for a small $K \in \mathbf{Set}^{\Delta^{\mathrm{op}}}$, such that $\bar q$ is a colimit diagram. Denote the restrictions by $p := \bar p|_K$ and $q := \bar q|_K$ and suppose that $\alpha := \bar\alpha|_K\colon p \to q$ is cartesian i.e. for every $f\colon x \to y$ in K, the square below is a pullback*

$$\begin{tikzcd}
p(x) \arrow[r, "p(f)"] \arrow[d, "\alpha"] & p(y) \arrow[d, "\alpha"] \\
q(x) \arrow[r, "q(y)"] & q(y).
\end{tikzcd}$$

Then, given the assumptions above, the condition is that

$$\bar p \text{ is a colimit} \quad\iff\quad \bar\alpha \text{ is cartesian.} \qquad (\boxed{\mathcal{R}})$$

Proof of Lem. 2.4.3. Using the standard characterisation of presentable categories [HTT] one concludes that $\mathcal{X}_{/U}$ is presentable, since it is accessible by [HTT, 5.4.6.7] and contains all colimits by [HTT, 1.2.13.8]. Thus, assume that there are $\bar p, \bar q\colon K\star\Delta^0 \to \mathcal{X}_{/U}$ and $\bar\alpha$ as in 2). It suffices to show that condition $\boxed{\mathcal{R}}$ holds. Let $t\colon U \to 1_{\mathcal{X}}$ be the unique morphism to the terminal object of $\mathcal{X}$. Consider the adjoint functors

$$\begin{tikzcd}
\mathcal{X} \arrow[r, shift right, "t^*"'] & \mathcal{X}_{/U} \arrow[l, shift right, "t_!"']
\end{tikzcd}$$

given by postcomposition and pullback. Composing with $t_!$ yields the diagrams

$$\begin{tikzcd}
K\star\Delta^0 \arrow[r, shift left, "\bar p"] \arrow[r, shift right, "\bar q"'] & \mathcal{X}_{/U} \arrow[r, "t_!"] & \mathcal{X}.
\end{tikzcd}$$

Since $t_!$ preserves colimits and pullbacks (see Prop. A.1.6), $t_!\circ\bar q$ is a colimit and $t_!\circ\alpha\colon t_!\circ p \to t_!\circ q$ is cartesian. Hence, the assumptions in 2) for $\mathcal{C} = \mathcal{X}$ are satisfied. Consequently, the fact, that $\mathcal{X}$ is an ∞-topos and $t_!$ detects pullbacks and colimits

$$\bar p \text{ is a colimit} \overset{\text{A.1.6}}{\iff} t_!\circ\bar p \text{ is a colimit} \overset{\mathcal{X}\ \infty\text{-topos}}{\iff} t_!\circ\bar\alpha \text{ is cartesian} \overset{\text{A.1.6}}{\iff} \bar\alpha \text{ is cartesian.} \qquad \square$$

Corollary 2.4.5. *Let $\mathcal{X}$ be an ∞-topos. The slicing functor $\mathcal{X}_{/-}\colon \mathcal{X} \to \widehat{\mathrm{Cat}}_\infty$ (see Lem. A.1.1) assigns each $f\colon X \to Y$ the functor $f_!$ of the essential geometric morphism*

$$\begin{tikzcd}
\mathcal{X}_{/X} \arrow[r, bend left, "f_!"] \arrow[r, bend right, "f_*"'] & \mathcal{X}_{/Y} \arrow[l, "f^*"']
\end{tikzcd}$$

Thus, $\mathcal{X}_{/-}$ factors through $\mathcal{RT}\mathrm{op}^{\mathrm{ess}}$ and is equivalent to functors

$$\begin{aligned} \mathcal{X}^*_{/-}\colon \mathcal{X}^{\mathrm{op}} \to \mathcal{LT}\mathrm{op}, &\qquad (f\colon X \to Y) \mapsto (\mathcal{X}_{/Y} \xrightarrow{f^*} \mathcal{X}_{/X}) \\ (\mathcal{X}_{/-})_*\colon \mathcal{X} \to \mathcal{RT}\mathrm{op}, &\qquad (f\colon X \to Y) \mapsto (f_*\colon \mathcal{X}_{/X} \xrightarrow{f_*} \mathcal{X}_{/Y}) \end{aligned}$$

Since adjunctions carry over to slice categories (cf.[HTT, 5.2.5.1]), so do geometric morphisms.

Corollary 2.4.6. *Let $g_*\colon \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of ∞-topoi and $f\colon X \to X'$ a morphism in $\mathcal{X}$. Slicing produces the following commutative squares in $\mathcal{RT}\mathrm{op}$ and $\mathcal{LT}\mathrm{op}$*

$$\begin{tikzcd} \mathcal{X}_{/X} \arrow[r, "f_*"] \arrow[d, "g_*"] & \mathcal{X}_{/X'} \arrow[d, "g_*"] \\ \mathcal{Y}_{/g_*(X)} \arrow[r, "(g_*(f))_*"] & \mathcal{Y}_{/g_*(X')} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{X}_{/X'} \arrow[r, "f^*"] \arrow[d, "g^*"] & \mathcal{X}_{/X} \arrow[d, "g^*"] \\ \mathcal{Y}_{/g^*(X')} \arrow[r, "(g^*(f))^*"] & \mathcal{Y}_{/g^*(X)} \end{tikzcd}$$

Definition 2.4.7. Let $X \in \mathcal{X}$ be an object in an ∞-topos and $t\colon X \to 1_{\mathcal{X}}$ the morphism to a terminal object. A geometric morphism in $\mathcal{RT}\mathrm{op}$ is called **étale**, if it is equivalent to the right adjoint $t_*\colon \mathcal{X}_{/X} \to \mathcal{X}$ of the essential geometric morphism of Cor. 2.4.5. The subcategory of $\mathcal{RT}\mathrm{op}_{/\mathcal{X}}$ étale morphisms with target $\mathcal{X}$ is denoted by $\mathcal{RT}\mathrm{op}^{ét}_{/\mathcal{X}}$ A geometric morphism in $\mathcal{LT}\mathrm{op}$ is étale, if it is equivalent to the pullback functor t^*.

Recall, that $\Gamma_{\mathcal{X}}$ is the functor $\mathrm{map}_{\mathcal{X}}(1_{\mathcal{X}},\text{-})$. Thus, $\Gamma_{\mathcal{X}}(X) \simeq \mathcal{X}_* \times_{\mathcal{X}} \{X\}$. In this sense the next Proposition, which is stated without proof, shows that $\mathcal{X}_{/X}$ classifies sections of X i.e. is universal among the $g^*\colon \mathcal{Y} \to \mathcal{X}$ in $\mathcal{LT}\mathrm{op}_{/\mathcal{X}}$, such that g^*X admits a global section.

Proposition 2.4.8 ([HTT, 6.3.5.5])**.** *Let $X \in \mathcal{X}$ be an object in an ∞-topos and let $1_X = id_X \in \mathcal{X}_{/X}$ be the terminal object. The étale geometric morphism $t^*\mathcal{X} \to \mathcal{X}_{/X}$ of Def. 2.4.7 together with the canonical map $\alpha\colon 1_X \to t^*(X) \simeq X \times_{1_{\mathcal{X}}} X$ exhibit $\mathcal{X}_{/X}$ as classifying ∞-topos for sections of X i.e. for every ∞-topos $\mathcal{Z}$ the following diagram is a pullback square*

$$\begin{tikzcd} \mathrm{Fun}^*(\mathcal{X}_{/X}, \mathcal{Z}) \arrow[r, "\circ t^*"] \arrow[d, "\mathrm{ev}_\alpha"'] & \mathrm{Fun}^*(\mathcal{X}, \mathcal{Z}) \arrow[d, "\mathrm{ev}_X"] \\ \mathcal{Z}_* \arrow[r, "\mathrm{ev}_1"] & \mathcal{Z}. \end{tikzcd}$$

Corollary 2.4.9 ([HTT, 6.3.5.6])**.** *Let $\mathcal{X}$ and $\mathcal{Z}$ be ∞-topoi, $U \in \mathcal{X}$ and $\mathcal{X} \xrightarrow{t^*} \mathcal{X}_{/U}$ the corresponding étale morphism. Then, $F\colon \mathrm{Fun}^*(\mathcal{X}_{/X}, \mathcal{Z}) \xrightarrow{\circ t^*} \mathrm{Fun}^*(\mathcal{X}, \mathcal{Z})$ is a left fibration, since $\mathcal{Z}_* \to \mathcal{Z}$ is a left fibration.*
Moreover, the fiber of F over any $g^* \in \mathrm{Fun}^*(\mathcal{X}, \mathcal{Z})$ is equivalent to $\mathrm{map}_{\mathcal{Z}}(1_{\mathcal{Z}}, g^*(X))$, since this is the fiber of $\mathcal{Z}_* \to \mathcal{Z}$ over $g^*(X)$.

Corollary 2.4.10. *Taking groupoid cores in Cor. 2.4.9 produces the fiber sequence*

$$\mathrm{map}_{\mathcal{Z}}(1_{\mathcal{Z}}, g^*(X)) \to \mathrm{map}_{\mathcal{LT}\mathrm{op}}(\mathcal{X}_{/X}, \mathcal{Z}) \to \mathrm{map}_{\mathcal{LT}\mathrm{op}}(\mathcal{X}, \mathcal{Z}). \qquad (\boxed{\mathfrak{F}})$$

In case $\mathcal{X} = \mathrm{Ani}$, the sequence $\boxed{\mathfrak{F}}$ implies that

$$\Gamma_{\mathcal{Z}}(\underline{K}) \simeq \mathrm{map}_{\mathcal{LT}\mathrm{op}}(\mathrm{Ani}_{/K}, \mathcal{X}) \qquad \forall \mathcal{Z} \in \mathcal{LT}\mathrm{op}, \forall K \in \mathrm{Ani}.$$

since Ani *is initial in* $\mathcal{LT}\mathrm{op}$.

Corollary 2.4.11. *In case* $\mathcal{Z} = \mathcal{X}_{/U}$ *for an object* $U \in \mathcal{X}$*, the fiber sequence* $\boxed{\mathfrak{F}}$ *yields*

$$\begin{aligned} \mathrm{map}_{\mathcal{X}}(U, X) &\overset{A.1.3}{\simeq} \mathrm{map}_{\mathcal{X}_{/U}}(\mathrm{id}_U, X \times U) \simeq \mathrm{map}_{\mathcal{LT}\mathrm{op}}(\mathcal{X}_{/X}, \mathcal{X}_{/U}) \times_{\mathrm{map}_{\mathcal{LT}\mathrm{op}}(\mathcal{X}, \mathcal{X}_{/U})} \{t_U^*\} \\ &\overset{A.1.2}{\simeq} \mathrm{map}_{\mathcal{LT}\mathrm{op}_{\mathcal{X}/}}(\mathcal{X}_{/X}, \mathcal{X}_{/U}). \end{aligned}$$

where $t_U^* \colon \mathcal{X} \to \mathcal{X}_{/U}$ *is geometric morphism obtained by pulling back along the unique map* $t_U \colon U \to 1_{\mathcal{X}}$ *to a terminal object. In consequence, the slicing functor* $\mathcal{X}_{/-}^* \colon \mathcal{X}^{\mathrm{op}} \to \mathcal{LT}\mathrm{op}_{\mathcal{X}/}$ *and its analogue* $(\mathcal{X}_{/-})_* \colon \mathcal{X} \to \mathcal{RT}\mathrm{op}_{/\mathcal{X}}$ *are both fully faithful.*

2.5 Open and Closed Immersions

For a topological space X with open subset $j \colon U \hookrightarrow X$ and closed complement $i \colon Z \hookrightarrow X$, a sheaf of abelian groups $\mathcal{F}$ can be reconstructed via the short exact sequence of [Har77, Ex. II.1.19]

$$0 \to j_! j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \to 0$$

In this section, we are going to see that the non-stable settings, we can still recover a fiber sequence of this form induced by the units of the adjunctions $j_! \dashv j^* \dashv j_*$ and $i^* \dashv i_*$. To do so, it is convenient to to define open and closed immersions of ∞-topoi.

Definition 2.5.1 ([BGH20, 3.2.2])**.** Let $\mathcal{X}$ be a ∞-topos. An object $U \in \mathcal{X}$ is **open** if it is a subobject of $1_{\mathcal{X}}$ i.e. if it is -1-truncated. Let $\mathrm{Op}(\mathcal{X}) := \tau_{\leq -1}\mathcal{X} \subset \mathcal{X}$ be the full subcategory of open objects.
A geometric morphism $g_* \colon \mathcal{X} \to \mathcal{Y}$ is an **open immersion**, if it is equivalent to the étale morphism $t_* \mathcal{X} \to \mathcal{X}_{/U}$ for an $U \in \mathrm{Op}(\mathcal{X})$.

As usual, closed immersions arise as complements of open objects.

Definition 2.5.2. Let $\mathcal{X}$ be an ∞-topos and $U \in \mathrm{Op}(\mathcal{X})$. An object $X \in \mathcal{X}$ is **trivial on** U if, whenever there is a map $\tilde{U} \to U$ in $\mathcal{X}$, then $\mathrm{map}_{\mathcal{X}}(\tilde{U}, X) \simeq *$. The full subcategory $\mathcal{X}/U \subset \mathcal{X}$ of objects, which are trivial on U, is called the **closed subtopos of $\mathcal{X}$ complementary to U**.

This full subcategory is indeed an ∞-topos

Lemma 2.5.3 ([HTT, 7.3.2.4])**.** *There is a class of monomorphisms* S_U *in* $\mathcal{X}$*, such that* $\mathcal{X}/U \subset \mathcal{X}$ *are precisely the* S_U*-local objects in the sense of Rmk. 2.2.15. Thus, there is a topological localization* $i^* \colon \mathcal{X} \to \mathcal{X}/U$*.*

Proposition 2.5.4. *For every* ∞*-topos* $\mathcal{X}$ *and every subobject* U *of a terminal object* $1_{\mathcal{X}}$*, the pair of morphisms*

$$\begin{tikzcd} \mathcal{X}_{/U} \arrow[r, shift right, "j_*"'] & \mathcal{X} \arrow[l, shift right, "j^*"'] \arrow[r, shift left, "i_*"] & \mathcal{X}/U \arrow[l, shift left, "i^*"] \end{tikzcd}$$

forms a recollement, that is

1) $j^* i_*$ *is equivalent to the terminal functor* $t \colon \mathcal{X}/U \to \Delta^0 \xrightarrow{1_U} \mathcal{X}_{/U}$*, which sends every object to the terminal object* $1_U \in \mathcal{X}_{/U}$*.*

2) j^* and i^* are jointly conservative, i.e., if $f : x \to y$ is a morphism in $\mathcal{X}$ such that j^*f and i^*f are equivalences, then f is an equivalence.

Note. For many applications it is convenient to define *recollements* in broader generality simply by relying on the conditions above (cf. A.5.1). Every such *recollement* gives rise to a pullback square of endofunctors

$$\begin{tikzcd}
\mathrm{id}_{\mathcal{X}} \arrow[r, "\eta_i"] \arrow[d, "\eta_j"'] & i_*i^* \arrow[d, "i_*i^*\bullet\eta_j"] \\
j_*j^* \arrow[r, "\eta_i\bullet j_*j^*"] & i_*i^*j_*j^*
\end{tikzcd}$$

where η_i and η_j are the units of the adjunctions $i^* \dashv i_*$ and $j^* \dashv j_*$. The argument given for this fact, also shows, that A.5.2

Remark 2.5.5. In [HA, A.8.15], Lurie shows, that every recollement for an ∞-topos $\mathcal{X}$ is of the form of Prop. 2.5.4.

Proof. 1) j^* admits a fully faithful left adjoint $j_! \colon \mathcal{X}_{/U} \to \mathcal{X}$, which sends an object $\tilde{U} \to U$ of $\mathcal{X}_{/U}$ to $\tilde{U}$. Consequently, j^*i_*Z is a terminal object for every $Z \in \mathcal{X}/U$, since by Definition of $\mathcal{X}/U$, we have for every $\tilde{U} \in \mathcal{X}_{/U}$:

$$\mathrm{map}_{\mathcal{X}_{/U}}(\tilde{U}, j^*i_*Z) \simeq \mathrm{map}_{\mathcal{X}}(j_!\tilde{U}, i_*Z) \simeq \mathrm{map}_{\mathcal{X}}(\tilde{U}, i_*Z) \simeq *.$$

2) This follows from the proof of A.5.2. ☐

Proposition 2.5.6. *In the situation of Prop. 2.5.4, there is a fiber sequence of endofunctors*

$$j_!j^* \to id_{\mathcal{X}} \to i_*i^*$$

where $j_!$ is the fully faithful left adjoint of the étale morphism $j^* \colon \mathcal{X} \to \mathcal{X}_{/U}$.

Before we start with the proof, notice that $j_!$ is precisely the functor that annihilates i^*.

Observation 2.5.7. In the situation of Prop. 2.5.4, the composition $i^*j_!$ is equivalent to the functor $\mathcal{X}_{/U} \to \mathcal{X}/U$ constant at the initial object $\emptyset_{\mathcal{X}/U}$, since for every $u \in \mathcal{X}_{/U}$ and $z \in \mathcal{X}/U$ we have

$$\mathrm{map}_{\mathcal{X}/U}(i^*j_!(u), z) \simeq \mathrm{map}_{\mathcal{X}_{/U}}(u, j^*i_*(z)) \simeq \mathrm{map}_{\mathcal{X}_{/U}}(u, 1_U) \simeq *$$

Thus, $i^*j_!(u)$ is an initial object.

Proof of Prop. 2.5.6. Fix an arbitrary $X \in \mathcal{X}$. Let F denote the pullback in the square induced by the counit ϵ_j of j and the unit of i:

$$\begin{tikzcd}
j_!j^*X \arrow[rrd, bend left, "\epsilon_j"] \arrow[rd, dashed, "s"] \arrow[rdd, bend right, "\eta_i"'] & & \\
& F \arrow[r] \arrow[d] & X \arrow[d, "\eta_i"] \\
& 1_{\mathcal{X}} \simeq j_!j^*i_*i^*X \arrow[r, "\epsilon_j \circ i_*i^*"] & i_*i^*X
\end{tikzcd}$$

By Prop. 2.5.4.2), it suffices to check, that s is an equivalence after applying i^* and j^*.
By Obs. 2.5.7, $i^*j_!j^*X$ is an initial object, but since $i^*X \xrightarrow{\simeq} i^*i_*i^*X$ is an equivalence, the fiber i^*F is an initial object as well.
Applying j^* on the other hand, the square becomes

$$\begin{tikzcd}
j^*j_!j^*X \arrow[r] \arrow[d] & j^*X \arrow[d] \\
1_{\mathcal{X}} \arrow[r] & j^*i_*i^*X \simeq 1_{\mathcal{X}}.
\end{tikzcd}$$

As both horizontal arrows are equivalences, this is a pullback square. Thus, $j^*F \simeq j^*j_!j^*X$. $\square$

2.6 Truncations and Localic ∞-Topoi

Truncations provide a fundamental mechanism to stratify objects within an ∞-topos regarding their the homotopical complexity. This notion permits the fully faithful embedding of classical $(n, 1)$-topoi into the ∞-categorical setting, identifying sheaf $(n, 1)$-topoi as ∞-sheaves with values in n-truncated anima and thereby streamlining many higher-categorical arguments. We follow [HTT, §5.5.6 & §6.4.5].

Definition 2.6.1. An object c of an ∞-category $\mathcal{C}$ is **k-truncated** for $k \geq -1$, if $\pi_n(\mathrm{map}_{\mathcal{C}}(b, c), x) \cong *$ for all $n > k$ and for every $b \in \mathcal{C}$ and base-point x. Let $\tau_{\leq k}\mathcal{C} \subset \mathcal{C}$ denote the full subcategory of k-truncated objects. An object $c \in \mathcal{C}$ is **truncated**, if it is k-truncated for some $k \geq -1$. Let $\tau_{<\infty}\mathcal{C} \subset \mathcal{C}$ be the full subcategory of truncated objects.

Notation 2.6.2. 0-truncated objects are also called discrete. There is a canonical equivalence $\tau_{\leq 0}\mathcal{C} \simeq \mathrm{N}(h(\tau_{\leq 0}\mathcal{C}))$, relating $\tau_{\leq 0}\mathcal{C}$ to its underlying 1-category, since equivalent morphisms in $\tau_{\leq 0}\mathcal{C}$ are already isomorphic. Abbreviate $\mathrm{Disc}(\mathcal{C}) := h(\tau_{\leq 0}\mathcal{C})$.

In presentable categories the truncated objects form a reflective subcategory. Precisely:

Proposition 2.6.3 ([HTT, 5.5.6.18])**.** *If $\mathcal{C}$ is a presentable ∞-category, the (fully faithful) inclusion $\tau_{\leq k}\mathcal{C} \hookrightarrow \mathcal{C}$ admits an accessible left adjoint $\tau_{\leq k} : \mathcal{C} \to \tau_{\leq k}\mathcal{C}$ for any $k \geq -1$.*

This functor commutes with morphisms of $\mathcal{LT}$op and limits.

Proposition 2.6.4 ([HTT, 5.5.6.28])**.** *For presentable categories $\mathcal{C}$, $\mathcal{D}$ and any left exact, presentable functor $F : \mathcal{C} \to \mathcal{D}$, there is an equivalence: $F \circ \tau_{\leq k} \simeq \tau_{\leq k} \circ F$.*

Lemma 2.6.5 ([HTT, 6.5.1.2])**.** *For any ∞-topos $\mathcal{X}$ and $n \geq 0$, the truncation functor $\tau_{\leq n} \colon \mathcal{X} \to \tau_{\leq n}\mathcal{X}$ preserves finite products.*

The notion of truncation is extended to morphisms in the following way.

Definition 2.6.6. Recall that a map $f \colon X \to Y$ of anima is k-truncated, if its homotopy fiber over any base point in Y is k-truncated. Then, a morphisms $f \colon c \to d$ in an ∞-category $\mathcal{C}$ is **k-truncated**, if $f^* \colon \mathrm{map}_{\mathcal{C}}(e, c) \to \mathrm{map}_{\mathcal{C}}(e, d)$ is k-truncated for any $e \in \mathcal{C}$.

Hence, a monomorphism in the sense of Def. 2.2.13 is precisely a -1-truncated morphism.

Observation 2.6.7. For any monomorphism $c \to d$ in $\mathcal{C}$ which admits pullbacks, the canonical map $c \times_d c \to d$ is an equivalence. This is, because $\operatorname{map}_{\mathcal{C}_{/d}}((c \times_d c \to d), (c \to d))$ is not empty.

Proposition 2.6.8 ([nLa25c, 4.2])**.** *Let $\mathcal{C}$ be a small ∞-category and $P \in \mathcal{P}(\mathcal{C})$. Then, $P \in \tau_{\leq n}\mathcal{P}(\mathcal{C})$ if and only if P factors through $\tau_{\leq n}\mathrm{Ani}$.*

Proof. If P is n-truncated, then $P(c) \simeq \operatorname{map}_{\mathcal{P}(\mathcal{C})}(y(c), P)$ is n-truncated for every $c \in \mathcal{C}$.

To prove the converse, suppose that all values of P are n-truncated and let $Q \in \mathcal{P}(\mathcal{C})$ be some presheaf. The description of presheaf as left Kan extensions gives

$$\operatorname{map}_{\mathcal{P}(\mathcal{C})}(Q, P) \overset{\text{[Wag23, 6.30]}}{\simeq} \operatorname{map}_{\mathcal{P}(\mathcal{C})}(\operatorname{colim}_{c \in \mathcal{C}_{/Q}}(y(c), P) \simeq \lim_{c \in \mathcal{C}_{/Q}} \operatorname{map}_{\mathcal{P}(\mathcal{C})}(y(c), P).$$

Since the right side is a limit of n-truncated anima, it is n-truncated, as well. ☐

Iteratively truncating an ∞-topos $\mathcal{X}$ gives rise to a Postikov tower style sequence

$$\mathcal{X} \to \cdots \to \tau_{\leq n+1}\mathcal{X} \to \tau_{\leq n}\mathcal{X} \to \tau_{\leq n-1}\mathcal{X} \to \cdots \to \tau_{\leq 0}\mathcal{X} \to \tau_{\leq -1}\mathcal{X}$$

An ∞-topos is n-localic if it models the n-th stage of such a Postikov tower. More precisely:

Definition 2.6.9 ([nLa25b, 2.1])**.** A ∞-topos $\mathcal{X}$ is **n-localic**, if the induced map $\mathrm{Fun}_*(\mathcal{Y}, \mathcal{X}) \xrightarrow{\tau_{\leq n-1}} \mathrm{Fun}_*(\tau_{n-1}\mathcal{Y}, \tau_{n-1}\mathcal{X})$ is an equivalence for any ∞-topos $\mathcal{Y}$.

Any n-category – regarded as an $n-1$-truncated ∞-category – gives rise to an n-localic ∞-topos. Quite remarkably, the converse is also true and reads as follows.

Proposition 2.6.10 ([HTT, Prop.6.4.5.9])**.** *An ∞-topos $\mathcal{X}$ is n-localic for $n \in \mathbb{N}$ if and only if there is a small n-site (C, τ) with all finite limits, such that $\mathrm{Shv}_\tau(C) \simeq \mathcal{X}$.*

Example 2.6.11. Given a small 1-site C, Prop. 2.6.8 identifies $\mathrm{Disc}(\mathcal{P}(\mathrm{N}(C)))$ with the 1-category $\mathcal{P}(C; \mathrm{Set})$. Furthermore, $\mathrm{Disc}(\mathrm{Shv}_\tau(\mathrm{N}(C))) \cong \mathrm{Shv}_\tau(C; \mathrm{Set})$, since truncation commutes with sheafification by Prop. 2.6.4.

When $\mathcal{X}$ arises from a 1-presite, [Sta25, Tag 03A0] gives explicit conditions under which the geometric morphism obtained in Thm. 2.3.23 is an equivalence.

Lemma 2.6.12 ([Sta25, Tag 03A0])**.** *Let $(\mathcal{C}, \sigma), (\mathcal{D}, \tau)$ be 1-presites and let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that:*

1) u is a morphism of presites in the sense of Def. 2.2.18.

2) u is cocontinuous i.e. for every $U \in \mathcal{C}$ and any covering $\{V_i \xrightarrow{f_i} u(U)\}_{i \in I}$ in $\mathcal{D}$ there exists a covering $\{U_i \xrightarrow{g_i} U\}_{i \in J}$, such that $\{u(U_i) \to u(U)\}_{i \in J}$ refines[21] $\{f_i\}_{i \in I}$.

3) Given $a, b : U' \to U$ in $\mathcal{C}$ such that $u(a) = u(b)$, there exists a covering $\{\, f_i : U'_i \to U' \,\}$ in $\mathcal{C}$ such that $a \circ f_i = b \circ f_i$.

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[21] i.e. there is map $\alpha J \to I$ and a morphism $u(U_j) \xrightarrow{\beta_j} V_{\alpha(j)}$ for each $j \in J$, such that $u(g_j) \cong f_{\alpha(j)} \circ \beta_j$.

4) Given $U', U \in \mathrm{Ob}(\mathcal{C})$ and a morphism $c : u(U') \to u(U)$ in $\mathcal{D}$, there exists a covering $\{ f_i : U'_i \to U' \}$ in $\mathcal{C}$ and morphisms $c_i : U'_i \to U$ such that $u(c_i) = c \circ u(f_i)$.

5) Given $V \in \mathrm{Ob}(\mathcal{D})$, there exists a covering of V in $\mathcal{D}$ of the form $\{ u(U_i) \to V \}_{i \in I}$.

Then, the morphism of 1-topoi $u^*\colon \mathrm{Shv}(\mathcal{C}; \mathrm{Set}) \to \mathrm{Shv}(\mathcal{D}; \mathrm{Set})$ is an equivalence.

Indeed, Prop. 2.6.10 implies that this equivalence extends to ∞-sheaves.

Corollary 2.6.13. *Under the assumptions of Lem. 2.6.12 the geometric morphism $u^*\colon \mathrm{Shv}(\mathcal{C}) \to \mathrm{Shv}(\mathcal{D})$ of Thm. 2.3.23 is an equivalence.*

Note. While this result is quite versatile, one might expect a similar statement to hold for ∞-categories directly. However, the strategy used in [Sta25, Tag 03A0] does not seem to generalize in a straightforward manner. A related result for "quasi-topologies" appears in [Hoy14, Lemma C.3], though it requires stronger assumptions, such as the full faithfulness of u.

Lem. 2.6.12 can be used to identify the ∞-topoi Ex. 2.2.17 with the usual version arising from the corresponding small sites.

Definition 2.6.14. If $(\mathcal{C}, \tau)$ is a (pre)site and $X \in \mathcal{C}$ an object, then the **small (pre)site** X_τ associated to τ is the full subcategory of $\mathcal{C}_{/X}$ spanned by those morphisms, which are part of some covering family of X, equipped with the (pre)topology τ.

Example 2.6.15. Let X be a topological space and let $\mathrm{Op}(X) \subset \mathrm{Top}_{/X})$ be category of open sets of X i.e. the small site corresponding to $(\mathrm{Top}_{/X}, \tau_{\mathrm{open}})$ (see Ex. 2.2.17). Similarly, let $\mathrm{lHomeo}(X) \subset \mathrm{Top}_{/X}$ be the small site corresponding to the pretopology τ_{lhomeo}, whose covering are surjective local homeomorphisms. The inclusions

$$\mathrm{Op}(X) \subset \mathrm{Top}_{/X} \qquad \text{and} \qquad \mathrm{lHomeo}(X) \subset \mathrm{Top}_{/X}$$

satisfy the hypothesis of Lem. 2.6.12 and thus induce equivalences

$$\mathrm{Shv}(\mathrm{Op}(X)) \simeq \mathrm{Shv}_{\mathrm{open}}(X) \qquad \text{and} \qquad \mathrm{Shv}(\mathrm{lHomeo}(X)) \simeq \mathrm{Shv}_{\mathrm{lhomeo}}(X).$$

The inclusion $\mathrm{Op}(X) \subset \mathrm{lHomeo}(X)$ likewise satisfies the conditions of Lem. 2.6.12, yielding

$$\mathrm{Shv}_{\mathrm{open}}(X) \simeq \mathrm{Shv}_{\mathrm{lhomeo}}(X).$$

Henceforth, $\mathrm{Op}(X)$ is adopted as the default site underlying $\mathrm{Shv}_{\mathrm{open}}(X)$.

Example 2.6.16. Let X be a scheme and let $X_{ét} \subset \mathrm{Sch}_{/X}$ be the small étale site i.e. the small site associated to the étale topology $\tau_{ét}$ (see § C.4). Again, the inclusion $X_{ét} \subset \mathrm{Sch}_{/X}$ satisfies the conditions of Lem. 2.6.12 and thus induces

$$\mathrm{Shv}(X_{ét}) \simeq \mathrm{Shv}_{ét}(X).$$

Let CRing denote the 1-category of commutative rings and set $\mathrm{Aff} := \mathrm{N}(\mathrm{CRing})^{\mathrm{op}}$. Then, the inclusion $(\mathrm{Aff}, \tau_{ét}) \hookrightarrow (\mathrm{Sch}, \tau_{ét})$ and the corresponding inclusion of small sites $(X_{\mathrm{Aff}}, \tau_{ét}) \hookrightarrow$

$(X_{\acute{e}t}, \tau_{\acute{e}t})$ both satisfy the assumptions of Lem. 2.6.12, yielding

$$\mathrm{Shv}_{\acute{e}t}(\mathrm{Aff}) \simeq \mathrm{Shv}_{\acute{e}t}(\mathrm{Sch}) \qquad \text{and} \qquad \mathrm{Shv}_{\acute{e}t}(X_{\mathrm{Aff}}) \simeq \mathrm{Shv}_{\acute{e}t}(X)$$

As usual, $X_{\acute{e}t}$ is adopted as the default site underlying $\mathrm{Shv}_{\acute{e}t}(X)$.

2.7 Application: Points and Stalks

The identifications of Ex. 2.6.15 and Lem. 2.3.25 permit an explicit description of the points of the ∞-topoi from Ex. 2.2.17 in terms of points of their underlying sites. These are defines as follows.

Definition 2.7.1. A **point** of an ∞-topos $\mathcal{X}$ is a geometric morphism $p_*\colon \mathrm{Ani} \to \mathcal{X}$. The left adjoint p^* of such p_* is itself frequently called a point.

In case $\mathcal{X}$ arises from a presite it is convenient to specialize of the notion of a point.

Definition 2.7.2. Let $(\mathcal{C}, \tau)$ be a presite. A **point** of $(\mathcal{C}, \tau)$ is a left exact functor $p\colon \mathcal{C} \to \mathrm{Ani}$, inducing an effective epimorphism[22] $\coprod_{i\in I} p(x_i) \twoheadrightarrow p(x)$ for every $x \in \mathcal{C}$ and every covering $\{x_i \to x\}_{i\in I}$.

Observation 2.7.3. By Thm. 2.3.23, the points of a presite $\mathcal{C}$ and $\mathrm{Shv}_\tau(\mathcal{C})$ coincide.
If $\mathcal{C}$ is 1-site, then $\mathrm{Shv}_\tau(\mathcal{C})$ is 1-localic and therefore, the points of $\mathcal{C}$ correspond precisely to the left exact functors $p\colon \mathcal{C} \to \mathrm{Set}$, which send coverings to surjections.

Observation 2.7.4. Let $(\mathcal{C}, \tau)$ be a site. A point $p\colon \mathcal{C} \to \mathrm{Ani}$ preserves all colimits, which exist in $\mathcal{C}$, since p is isomorphic to the following composition of colimit preserving functors

$$\mathcal{C} \xrightarrow{y} \mathcal{P}(\mathcal{C}) \xrightarrow{L_\tau} \mathrm{Shv}_\tau(\mathcal{C}) \xrightarrow{P} \mathrm{Ani}$$

where P is the geometric morphism induced by p via Thm. 2.3.23.

Although this notion might seem fairly abstract, it relates to points in the classical sense.

Example 2.7.5. Let T be a topological space. A point $t \in T$ corresponds to a morphism $t\colon * \to T$. However, $\mathrm{N}(\mathrm{Op}(*)) \simeq \Delta^0$ and therefore, $\mathrm{Shv}_{\mathrm{open}}(*) \simeq \mathcal{P}(*) \simeq \mathrm{Ani}$. Hence, every point $t \in T$ produces the following point of the site

$$p_x\colon\ \mathrm{Op}(X) \xrightarrow{-\times_T t} \mathrm{Op}(*) \xrightarrow{y} \mathcal{P}(*) \xrightarrow{\simeq} \mathrm{Ani},\ U \mapsto \begin{cases} * \text{ if } x \in U \\ \emptyset \text{ if } x \notin U. \end{cases}$$

Example 2.7.6. Let X be a scheme and $\bar{x}$ a geometric point of X i.e. a morphism $\bar{x}\colon \mathrm{Spec}(K) \to X$ with K algebraically closed. Taking global sections induces equivalences

$$\mathrm{Shv}_{\acute{e}t}(\mathrm{Spec}\, K) \simeq \mathcal{P}(\mathrm{Spec}(K)_{\acute{e}t}) \simeq \mathrm{Ani}$$

Indeed, $\Gamma\colon \mathrm{Shv}_{\acute{e}t}(\mathrm{Spec}(K)) \to \mathrm{Ani}$ is clearly essentially surjective[23]. To verify fully faithfulness,

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[22] i.e. a surjection on connected components.

[23] Simply consider the constant sheaves.

let $F, G \in \mathcal{P}(\mathrm{Spec}(K)_{\acute{e}t})$. Describing F as colimit of representables yields

$$\mathrm{map}_{\mathcal{P}(\mathrm{Spec}(K)_{\acute{e}t})}(F,G) \simeq \lim_{Z \in \mathrm{Spec}(K)_{\acute{e}t/F}} \mathrm{map}_{\mathcal{P}(\mathrm{Spec}(K)_{\acute{e}t})}(y(Z),G) \simeq \lim_{Z \in \mathrm{Spec}(K)_{\acute{e}t/F}} \Gamma(G).$$

since every $Z \in \mathrm{Spec}(K)_{\acute{e}t}$ is a disjoint union of copies of $\mathrm{Spec}(K)$ (see Ex. C.4.9). Consequently, the above limit is equivalent to $\lim_{\Gamma(F)} \Gamma(G) \overset{\text{A.3.3}}{\cong} \Gamma(G)^{\Gamma(F)} \cong \mathrm{map}_{\mathrm{Ani}}(\Gamma(F), \Gamma(G)$ since:

$$\Gamma_{\mathcal{P}(\mathrm{Spec}(K)_{\acute{e}t})}(F) \simeq \mathrm{map}_{\mathcal{P}(\mathrm{Spec}(K))}(y(\mathrm{Spec}(K),F) \simeq \{y(\mathrm{Spec}(K))\} \times_{\mathcal{P}(\mathrm{Spec}(K)_{\acute{e}t})} \mathcal{P}(\mathrm{Spec}(K)_{\acute{e}t})^{/F}.$$

by [Lan21, 2.5.31]. In sum, every geometric point $\bar{x}\colon \mathrm{Spec}(K) \to X$ gives rise to a point of $X_{\acute{e}t}$, given by

$$F_{\bar{x}} := \bar{x}^*\colon X_{\acute{e}t} \xrightarrow{-\times_X \bar{x}} \mathrm{Spec}(K)_{\acute{e}t} \xrightarrow{y} \mathcal{P}(\mathrm{Spec}(K)_{\acute{e}t}) \simeq \mathrm{Ani}, \quad U \mapsto |U_{\bar{x}}|,$$

where $|U_{\bar{x}}|$ denotes the underlying set of $U_{\bar{x}} := U \times_X \mathrm{Spec}(K)$.

The Kan extension formula allows for the explicit calculation of stalks for such points.

Lemma 2.7.7. *Let $(\mathcal{C}, \tau)$ be a site and $p \in \mathcal{C}$ an object, such that* $\mathrm{Shv}(\mathcal{C}_{/p}) \simeq \mathrm{Ani}$. *For any morphism $f\colon p \to X$ in $\mathcal{C}$ and any $F \in \mathrm{Shv}_\tau(\mathcal{C}_{/X})$ the value of the inverse image $f^*(F)$ at $d \in \mathcal{C}_{/p}$ is given by*

$$(f^*F)(d) \simeq \operatorname*{colim}_{c \in ((\mathcal{C}_{/X})_{f_!(d)/})^{\mathrm{op}}} F(c)$$

where $f_!\colon \mathcal{C}_{/p} \to \mathcal{C}_{/X}$ denotes the postcomposition with f.

Proof. This follows from the pointwise left Kan extension formula (c.f. [Cho22, 4.4.1]). □

Definition 2.7.8. In the situation of Lem. 2.7.7, the composite

$$\mathrm{Shv}_\tau(\mathcal{C}_{/X}) \xrightarrow{f^*} \mathrm{Shv}_\tau \xrightarrow{\Gamma_{\mathrm{Shv}_\tau(\mathcal{C}_{/p})}} \mathrm{Ani}$$

is called the **stalk functor** associated to $f\colon p \to X$. For any $F \in \mathrm{Shv}_\tau(\mathcal{C}_{/X})$, the resulting anima $\Gamma_{\mathrm{Shv}_\tau(\mathcal{C}_{/p})}(f^*F)$ is called the **stalk** of F at p.

Example 2.7.9. Let $\bar{x}\colon \mathrm{Spec}(K) \to X$ be a geometric point of a scheme X. By Ex. 2.7.6, $\bar{x}$ satisfies the conditions of Lem. 2.7.7. Hence, the associated point $\bar{x}^*\colon \mathrm{Shv}_{\acute{e}t}(X) \to \mathrm{Ani}$ is described by the following formula for every $F \in \mathrm{Shv}_{\acute{e}t}(X)$

$$\bar{x}^*F \simeq \operatorname*{colim}_{(u\colon \mathrm{Spec}(K) \to U) \in (X_{\acute{e}t})_{\mathrm{Spec}(K)/}} F(U)$$

For simplicity, the slice-category in the index is abbreviated by $\mathrm{Nb}_{\acute{e}t}(\bar{x}) := (X_{\acute{e}t})_{\mathrm{Spec}(K)/}$. The objects $u\colon \mathrm{Spec}(K) \to U$ of this category are called étale neighbourhoods of $\bar{x}$.

2.8 Internal Homotopy Groups

The fact that any ∞-topos $\mathcal{X}$ is cotensored over Ani (cf. Lem. A.3.3) allows for a natural definition of homotopy groups internal to $\mathcal{X}$.

Definition 2.8.1. Consider the n-sphere $S^n = \delta\Delta^n \in \mathrm{Ani}$ and let $i_n \colon * \to S^n$ be a choice of basepoint for each $n \geq 0$. For any $X \in \mathcal{X}$, the map i_n induces a morphism in $\mathcal{X}$

$$^{X}i_n^* \colon X^{S^n} \to X.$$

Regarding $^{X}i_n^*$ as an object of $\mathcal{X}_{/X}$, the **n-th homotopy group** is defined as

$$\pi_n(X) := \tau_{\leq 0}(^{X}i_n^*) \in \tau_{\leq 0}\mathcal{X}_{/X}.$$

It is convenient to extend this Definition to morphisms $f \colon X \to Y$ in $\mathcal{X}$. Regarding them as objects in the ∞-topos $\mathcal{X}_{/Y}$ one sets

$$\pi_n(f) := \tau_{\leq 0}(^{f}i_n^*) \in \tau_{\leq 0}((\mathcal{X}_{/Y})_{/X}) \simeq \tau_{\leq 0}(\mathcal{X}_{/f}).$$

As a consequence of Lemma 2.6.5, homotopy groups commute with finite products.

Corollary 2.8.2. *For objects $X, Y \in \mathcal{X}$ there is an equivalence in $\tau_{\leq 0}\mathcal{X}_{/X\times Y}$*

$$\pi_n(X \times Y) \simeq \pi_n(X) \times \pi_n(Y) \qquad\qquad \forall\, n \geq 0$$

As this result is not explicitly recorded in [HTT], a proof is provided for the reader's convenience.

Proof. For $n = 0$ this is precisely Lem. 2.6.5. Denote the geometric morphisms induced by the projections from $X \times Y$ by $\mathrm{pr}_X^* \colon \mathcal{X}_{/X} \to \mathcal{X}_{/X\times Y}$ and $\mathrm{pr}_Y^* \colon \mathcal{X}_{/Y} \to \mathcal{X}_{/X\times Y}$.
Recall, that the product in $\mathcal{X}_{/X\times Y}$ is given by taking the pullback over $X \times Y$. Then, for the product of X^{S^n} and Y^{S^n} considered as objects in $\mathcal{X}_{/X}$ resp. $\mathcal{X}_{/Y}$ one calculates

$$\begin{aligned}
&\mathrm{pr}_X^*(X^{S^n}) \times_{X\times Y} \mathrm{pr}_Y^*(Y^{S^n}) = (X^{S^n} \times_X X \times Y) \times_{X\times Y} (Y^{S^n} \times_Y X \times Y) \\
&\simeq (X^{S^n} \times Y) \times_{X\times Y} (X \times Y^{S^n}) \simeq X^{S^n} \times Y^{S^n} \\
&\overset{2.3.28}{\simeq} \underline{\mathrm{Hom}}_{\mathcal{X}}(\underline{S^n}, X) \times \underline{\mathrm{Hom}}_{\mathcal{X}}(\underline{S^n}, Y) \simeq \underline{\mathrm{Hom}}_{\mathcal{X}}(\underline{S^n}, X \times Y) \overset{2.3.28}{\simeq} (X \times Y)^{S^n}.
\end{aligned}$$

since $\underline{\mathrm{Hom}}$ commutes with finite limits (cf. A.2.3). In consequence,

$$\mathrm{pr}_X^*(\pi_n(X)) \times \mathrm{pr}_Y^*(\pi_n(Y)) \overset{2.6.4}{\simeq} \mathrm{pr}_X^*(\tau_{\leq 0}X^{S^n}) \times_{X\times Y} \mathrm{pr}_Y^*(\tau_{\leq 0}Y^{S^n}) \overset{2.6.5}{\simeq} \tau_{\leq 0}(X \times Y)^{S^n} = \pi_n(X \times Y).$$

□

The next Lemma allows to interpret truncations in terms of vanishing of homotopy groups.

Lemma 2.8.3 ([HTT, 6.5.1.7])**.** *Let $n \geq 0$. An n-truncated morphism $f \colon X \to Y$ in $\mathcal{X}$ has $\pi_k(f) \simeq *$ for $k > n$. Moreover, this f is k-truncated for $-1 \leq k < n$ iff $\pi_{k+1}(f) \simeq *$.*
In particular, any n-truncated object $X \in \mathcal{X}$ has $\pi_k(X) \simeq *$ for $k > n$.

This leads to the notion of n-connected maps, which act as counterpart to n-truncated maps.

Definition 2.8.4 ([HTT, 6.5.1.10])**.** A morphism $f \colon X \to Y$ in $\mathcal{X}$ is n-connected, if it is an effective epimorphism and $\pi_k(f) \simeq *$ for $0 \leq k \leq n$. An $X \in \mathcal{X}$ is n-connected, if the unique map

$X \to 1_{\mathcal{X}}$ to the terminal object $1_{\mathcal{X}}$ is n-connected. In case $n = 0$, '0-connected' is abbreviated by 'connected'.

Note. Lurie [HTT] uses the term '$n+1$-connective' instead of 'n-connected'.

Proposition 2.8.5 ([HTT, 5.2.8.16])**.** *For any $-2 \leq n < \infty$, the class of n-connected and the class of n-truncated morphisms in $\mathcal{X}$ form "orthogonal factorization" system. In particular, every morphism $f\colon X \to Y$ in $\mathcal{X}$ factors as a n-connected followed by a n-truncated morphism.*

Corollary 2.8.6. *For any object $X \in \mathcal{X}$ and any $k \geq 0$, the unit map $X \to \tau_{\leq k} X$ is k-connected, since $\tau_{\leq k} X \to 1_{\mathcal{X}}$ is k-truncated.*

Lemma 2.8.7 ([HTT, 6.5.1.16/(5)])**.** *n-connected maps are stable under pullback for any $n \geq 0$.*

The homotopy groups $\pi_n(X)$ of an $X \in \mathcal{X}$ reside in the slice topos $\tau_{\leq 0}\mathcal{X}_{/X}$. Since working within $\mathcal{X}_{/X}$ can be technically cumbersome, the next Lemma proves particularly useful.

Lemma 2.8.8 ([HTT, 7.2.1.13])**.** *Let $\mathcal{X}$ be an ∞-topos, $n \geq 0$ and X an n-connected object of $\mathcal{X}$. The pullback functor $f^* : \mathcal{X} \to \mathcal{X}_{/X}$ restricts to the fully faithful functor $\tau_{\leq n} f^* \colon \tau_{\leq n}\mathcal{X} \to \tau_{\leq n}\mathcal{X}_{/X}$ which restricts to an equivalence from $\tau_{\leq n-1}\mathcal{X}$ to $\tau_{\leq n-1}\mathcal{X}_{/X}$.*
Moreover, the inverse of $\tau_{\leq n} f^*$ restricted to its essential image is the forgetful functor $f_!$.

Corollary 2.8.9. *For any $X \in \mathcal{X}$, consider $f^*\colon \tau_{\leq 0}\mathcal{X} \to \tau_{\leq 0}\mathcal{X}_{/X}$. If X is 1-connected, this map is an equivalence. Hence, for $n \geq 2$, every homotopy group $\pi_n(X) \in \tau_{\leq 0}\mathcal{X}_{/X}$ corresponds to a unique discrete abelian group $A \in \tau_{\leq 0}\mathcal{X} = \mathrm{N}(\mathrm{Disc}(\mathcal{X}))$.*

Prop. 2.6.4 also implies that homotopy groups commute with morphisms in $\mathcal{LT}\mathrm{op}$.

Corollary 2.8.10 ([HTT, 6.5.1.4])**.** *Let $f^* : \mathcal{Y} \to \mathcal{X}$ a geometric morphism in $\mathcal{LT}\mathrm{op}$. Then, $f^*\pi_n(Y) \simeq \pi_n(f^*Y)$ for every $Y \in \mathcal{Y}$.*
In particular, $f^*\pi_n(g) \simeq \pi_n(f^*g)$ for any morphism $g\colon Y \to Y'$ in $\mathcal{Y}$.

Proof. The second assertion is a special case of the first, by considering the geometric morphism $\mathcal{Y}_{/Y'} \to \mathcal{X}_{/f^*(Y')}$ instead of f^*. Now, the claimed equivalence is obtained as follows.

$$\begin{aligned} f^*\pi_n(Y) = f^*\tau_{\leq 0}(Y^{S^n} \to Y) &\overset{2.6.4}{\simeq} \tau_{\leq 0}(f^*(Y^{S^n}) \to f^*(Y)) \\ \overset{2.3.28,\ 2.3.29}{\simeq} &\quad \tau_{\leq 0}(f^*(Y)^{S^n} \to f^*(Y)) = \pi_n(f^*Y). \end{aligned}$$

Indeed, Cor. 2.3.29 applies, since S^n is a finite anima and f^* commutes with finite limits. ☐

2.9 Coherent ∞-Topoi

Coherent ∞-topoi provide a generalization of coherent spaces, which are precisely the compact, sober[24] spaces whose collection of compact subsets are closed under finite intersections and forms a basis for the topology of X. By a result of Hochster [Hoc69], these as precisely the spectral spaces i.e. homeomorphic to $\mathrm{Spec}(A)$ for an $A \in \mathrm{CRing}$. Identifying a space X with the terminal

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[24] A space X is called sober if for every non-empty, closed and irreducible subset $C \subset X$, there is a point $x \in X$ such that $C = \overline{\{x\}}$. See 7.3.4 for details.

object of $\mathrm{Op}(X)$ and intersections with the products in $\mathrm{Op}(X)$, the conditions for coherence are extended inductively to ∞-topoi as follows.

Definition 2.9.1 ([SAG, A.2.0.12, A.2.1.6])**.** Let $\mathcal{X}$ be an ∞-topos. ***n*-coherence** is defined in an inductive manner starting with

- $\mathcal{X}$ is **quasi-compact** or, equivalently **0-coherent**, if for every effective epimorphism $\coprod_{i\in I} U_i \to 1_{\mathcal{X}}$ there is a finite subset $I_0 \subset I$, such that $\coprod_{i\in I_0} U_i \to 1_{\mathcal{X}}$ is an effective epimorphism.

Once n-coherent ∞-topoi are defined on a level $n \geq 0$, this notion is specialised as follows

- An object $U \in \mathcal{X}$ is said to be ***n*-coherent**, if $\mathcal{X}_{/U}$ is n-coherent.
- $\mathcal{X}$ is **locally *n*-coherent**, if every $X \in \mathcal{X}$ admits an effective epimorphism $\coprod_{i\in I} U_i \to X$, where each U_i is n-coherent.

Relying on these specialisations, one proceeds with the induction step and sets

- $\mathcal{X}$ is **n + 1-coherent** if $\mathcal{X}$ is locally n-coherent and the collection of n-coherent objects is closed under finite products

Lastly, $\mathcal{X}$ is said to be **∞-coherent**, or simply **coherent**, if $\mathcal{X}$ is n-coherent for all $n \geq 0$. Likewise, an $X \in \mathcal{X}$ is **coherent** if $\mathcal{X}_{/X}$ coherent. Analogously, $\mathcal{X}$ is said to be **locally coherent** if every object $X \in \mathcal{X}$ admits an effective epimorphism $\coprod_{i\in I} U_i \to X$, where each U_i is coherent.

Observation 2.9.2. If X is a coherent topological space, then $\mathrm{Shv}_{\mathrm{open}}(X)$ is 1-coherent.

A good deal of the coherent ∞-topoi encountered in practice arise from Grothendieck sites, where each covering sieve is finite generated. Those are formalized as follows.

Definition 2.9.3 ([SAG, A.3.1.1])**.** A site $(\mathcal{C}, \sigma)$ is **finitary**, if for every $x \in C$ and every covering sieve $S \subset \mathcal{C}_{/x}$ there is a finite collection of objects $\{x_i \to x\}_{1\leq i\leq n} \subset S$, which generates a covering sieve $S_1 \subset S$.

Observation 2.9.4. If $(\mathcal{C}, \tau)$ is a presite, such that for every $x \in \mathcal{C}$ any covering $\{x_i \to x\}_{i\in I} \in \mathrm{Cov}_\tau(x)$ admits a finite subcovering[25] then τ generates a finitary Grothendieck topology

Finitary sites serve as a robust source for coherent ∞-topoi in the following way.

Proposition 2.9.5 ([SAG, A.3.1.3])**.** *Let $(\mathcal{C}, \tau)$ be a finitary site. Then,*

1) *The composition $L \circ y \colon \mathcal{C} \to \mathrm{Shv}_\tau(\mathcal{C})$ of the localization with the Yoneda embedding carries each object $C \in \mathcal{C}$ to a coherent object of $\mathrm{Shv}_\tau(\mathcal{C})$.*
2) *The ∞-topos $\mathrm{Shv}_\tau(\mathcal{C})$ is locally coherent.*
3) *If $\mathcal{C}$ has a final object, then $\mathrm{Shv}_\tau(\mathcal{C})$ is coherent.*

Definition 2.9.6. Let $\mathcal{X}$ be a coherent ∞-topos. Let $\mathcal{X}^{\mathrm{tc}} \subset \mathcal{X}$ denote the full subcategory of **truncated and coherent** objects.

---

[25] i.e. there is a finite subset $I_0 \subset I$ such that $\{x_i \to x\}_{i\in I_0} \in \mathrm{Cov}_\tau(x)$

Warning 2.9.7. Lurie [DAGXIII] refers to truncated coherent objects as 'finitely constructible'. Here, 'finitely constructible' will have a weaker meaning (see Def. 7.1.10). As noted in Prop. 7.1.18, the two notions coincide in spectral settings. However, it is unclear, whether they also coincide in non-spectral settings such as Berkovich Geometry.

Example 2.9.8. The truncated coherent objects in Ani are precisely the π-finite spaces.

The following results exhibit the stability of n-coherence under connected maps.

Proposition 2.9.9 ([SAG, A.2.4.1])**.** *Let $n \geq 0$ and $f\colon X \to Y$ be a morphism in $\mathcal{X}$. If X is n-coherent and f is $n-1$ connected, then Y is n-coherent, as well.*

A statement, similar to a converse to this result (see [SAG, A.2.4.1/(b_n)]) implies

Corollary 2.9.10 ([SAG, A.2.4.4])**.** *Suppose $\mathcal{X}$ is a coherent ∞-topos. For any n-coherent object $X \in \mathcal{X}$ for $n \geq 0$, the truncation $\tau_{n-1}X$ is a coherent object.*

Relying on these two facts, one may deduce the stability of coherence under homotopy groups

Corollary 2.9.11. *For any ∞-topos $\mathcal{X}$ and any coherent object $X \in \mathcal{X}$, each homotopy group $\pi_n X$ maps to a coherent object under the forgetful-map $f_!\colon \tau_{\leq 0}\mathcal{X}_{/X} \to \tau_{\leq 0}\mathcal{X}$.*

Proof. Since X is coherent, $id_X \in \mathcal{X}_{/X}$ is coherent and therefore $\pi_0(X) = \tau_{\leq 0} id_X \in \mathcal{X}_{/X}$ is coherent by Cor. 2.9.10. For $n \geq 1$, the morphism $X^{S^n} \to X$ admits a canonical section $s\colon X \to X^{S^n}$ over X, induced by the basepoint of S^n chosen in Def. 2.8.1. By the tensor-cotensor adjunction (see Def. A.3.1 and Lem. 2.3.28), this section corresponds to an unique morphism $s\colon \underline{S^n} \times X \to X$. Since homotopy groups commute with products (Lem. 2.8.2) and maps in $\mathcal{LT}$op (Lem. 2.6.4) one calculates

$$\pi_k(\underline{S^n} \times X) \simeq \pi_k(X) \times \underline{\Delta}_{\mathcal{X}}\, \pi_k(S^n) \simeq \pi_k(X) \quad \forall 0 < k < n.$$

Consequently, $X \times S^n \to X$ and $X \to X^{S^n}$ are $n-1$-connected. This makes X^{S^n} an n-coherent object by Prop. 2.9.9. As $n \leq 1$, Cor. 2.9.10 implies that $\tau_{\leq 0} X^{S^n}$ is an coherent object in $\mathcal{X}_{/X}$. Thus, it is coherent in $\mathcal{X}$, since $(\mathcal{X}_{/X})_{/X^{S^n}} \simeq \mathcal{X}_{/X^{S^n}}$. ☐

2.10 Hypercompleteness

Next, we turn to the notion of hypercompleteness, which provides a bridge to the classical theory of simplicial presheaves, as the standard model structure therein does not model general ∞-topoi, but only their hypercompletions (cf. [HTT, 6.5.3.13]). Conceptually, an ∞-topos is hypercomplete if it satisfies a version of the Whitehead Theorem.

Definition 2.10.1 ([HTT, 6.5.2.10])**.** An ∞-topos $\mathcal{X}$ is said to be **hypercomplete** if every ∞-connected morphism in $\mathcal{X}$ is an equivalence.

Any ∞-topos admits a hypercomplete sub-∞-topos consisting of its hypercomplete objects.

Definition 2.10.2 ([HTT, 6.5.2.8])**.** A object $X \in \mathcal{X}$ is **hypercomplete**, if it is local with respect to ∞-connected morphisms i.e. every ∞-connected map $Y \to Z$ in $\mathcal{X}$ induces an equivalence $\mathrm{map}_{\mathcal{X}}(Z, X) \simeq \mathrm{map}_{\mathcal{X}}(Y, X)$. Let $\mathcal{X}^{\mathrm{hyp}} \subset \mathcal{X}$ denote the full subcategory of hypercomplete objects.

Observation 2.10.3. $\mathcal{X}$ is hypercomplete iff $\mathcal{X} = \mathcal{X}^{\mathrm{hyp}}$. By Definition, $\mathcal{X}^{\mathrm{hyp}}$ is the localization $\mathcal{X}[\infty\text{-connected}]^{-1}$ of $\mathcal{X}$ at the collection of ∞-connected morphisms. Denote the localization functor – called **hypercompletion** – by $\mathfrak{h}\mathrm{yp}\colon \mathcal{X} \to \mathcal{X}^{\mathrm{hyp}}$. Since ∞-connected morphisms are stable under pullback (see Prop. 2.8.5), $\mathfrak{h}\mathrm{yp}$ is left exact by [HTT, 6.2.1.1] and therefore $\mathcal{X}^{\mathrm{hyp}}$ forms an ∞-topos.

As a localization, $\mathfrak{h}\mathrm{yp}$ satisfies the following universal property.

Proposition 2.10.4 ([HTT, 6.5.2.13])**.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be ∞-topoi, where $\mathcal{Y}$ is hypercomplete. Then, postcomposition with $\mathcal{X}^{\mathrm{hyp}} \subset \mathcal{X}$ induces an equivalence*

$$\mathrm{Fun}_*(\mathcal{Y}, \mathcal{X}^{\mathrm{hyp}}) \simeq \mathrm{Fun}_*(\mathcal{Y}, \mathcal{X}).$$

As a result of Prop. 2.8.5, every truncated object is local wrt. ∞-connected maps.

Proposition 2.10.5 ([HTT, 5.5.6.16, 6.5.2.9])**.** *For any ∞-topos $\mathcal{X}$ hypercompletion restricts to an equivalence $\mathfrak{h}\mathrm{yp}\colon \tau_{<\infty}\mathcal{X} \xrightarrow{\simeq} \tau_{<\infty}\mathcal{X}^{\mathrm{hyp}}$ on the truncated objects of $\mathcal{X}$ and $\mathcal{X}^{\mathrm{hyp}}$.*

In sufficiently good situations, hypercompleteness relates to coherence in the following way.

Proposition 2.10.6 ([SAG, A.2.2.2])**.** *Let $\mathcal{X}$ be an ∞-topos, which is locally n-coherent for all $n \geq 0$ (e.g. a coherent or locally coherent ∞-topos). Then*

1) $\mathcal{X}^{\mathrm{hyp}}$ is locally n-coherent for all $n \geq 0$.

2) The inclusion $\mathcal{X}^{\mathrm{hyp}} \subset \mathcal{X}$ preserves coherent objects.

3) An $X \in \mathcal{X}$ is coherent if and only if its hypercompletion $\mathfrak{h}\mathrm{yp}(X) \in \mathcal{X}^{\mathrm{hyp}}$ is coherent.

In particular, if $\mathcal{X}$ is locally coherent, then $\mathcal{X}^{hyp}$ is coherent.

From Prop. 2.10.5 and *2)* and *3)* above one retrieves the following equivalence.

Corollary 2.10.7. *Let $\mathcal{X}$ be an ∞-topos, which is locally n-coherent for all $n \geq 0$. Then, hypercompletion induces an equivalence*

$$\mathfrak{h}\mathrm{yp}\colon \mathcal{X}^{\mathrm{tc}} \to (\mathcal{X}^{\mathrm{hyp}})^{\mathrm{tc}}$$

between the truncated coherent objects of $\mathcal{X}$ and $\mathcal{X}^{\mathrm{hyp}}$.

A common practical criterion for hypercompleteness relies the condition, that an ∞-topos $\mathcal{X}$ can be fully probed by its points.

Definition 2.10.8. $\mathcal{X}$ is said to have **enough points**, if a morphism $f\colon X \to Y$ in $\mathcal{X}$ is an equivalence if and only if $q^*\colon q^*X \to q^*Y$ is an equivalence in Ani for all points q of $\mathcal{X}$.

Lemma 2.10.9. *If an ∞-topos has enough points, then it is hypercomplete.*

Proof. Suppose $\mathcal{X}$ has enough points and let $f\colon X \to Y$ be an ∞-connected morphism in $\mathcal{X}$. Since maps in $\mathcal{LT}\mathrm{op}$ preserve homotopy groups, the map $q^*f\colon q^*X \to q^*Y$ under any point $q_*\colon \mathrm{Ani} \to \mathcal{X}$ is an ∞-connected morphism in Ani. Then, q^*f is an equivalence in Ani by the Whitehead Theorem (see [Die10, 8.4.3]). Thus, f is an equivalence. $\square$

The properties of objects of an ∞-topos with enough points are determined by their stalks. This motivates the following definition.

Definition 2.10.10. Let p be a prime and let $\mathcal{X}$ be an ∞-topos with enough points. An object $X \in \mathcal{X}$ has **π-finite, p-coprime stalks**, if for every point $q^*\colon \mathcal{X} \to \mathrm{Ani}$, the anima q^*X is π-finite and p-coprime (Def. 1.0.3).

In accordance with the view that hypercomplete ∞-topoi represent the appropriate setting for classical results, the following theorem identifies hypercompleteness as a necessary condition (cf. Lemma 2.10.9) for extending Deligne's classical theorem (see [MM12, IX.11.3]) to the higher-categorical context.

Theorem 2.10.11 (Deligne-Lurie Completeness, [SAG, A.4.0.5])**.** *If an ∞-topos is locally coherent and hypercomplete, then it has enough points.*

2.11 Values in Presentable Categories

The sheaves considered thus far have been exclusively valued in anima. However, Ani serves as the universal category of values for higher sheaves. For example, the Dold-Kan correspondence [Wei94, 8.4.1] identifies the non-negatively graded chain complexes $\mathrm{Ch}_+(A)$ in an abelian category A with the category of simplicial objects $A^{\Delta^{\mathrm{op}}}$ in A. In case $A = \mathrm{Ab}$, every simplicial group object is a Kan simplicial object ([Wei94, 8.2.8]), results about $\mathrm{Shv}_\tau(\mathcal{C})$ for a site $(\mathcal{C}, \tau)$ specialise to statements for $\mathrm{Shv}_\tau(\mathcal{C}, \mathrm{Ch}_+(\mathrm{Ab}))$. Indeed, this is not a special case. The transition from $\mathrm{Shv}_\tau(\mathcal{C})$ to $\mathrm{Shv}_\tau(\mathcal{C}; \mathcal{D})$, for any presentable category $\mathcal{D}$, is functorial and obtained by "tensoring" with $\mathcal{D}$. This construction preserves many structural properties, underscoring the foundational role of anima-valued sheaves. As these generalities are not strictly required for the core arguments of this thesis, the following treatment is very brief. A more detailed exhibition, including proofs for all statements listed here can be found in [NA, §4.1], [HA] or [Vol24].

Proposition 2.11.1. *For any presentable categories $\mathcal{C}$ and $\mathcal{D}$ the full subcategories*

$$\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D}) \subset \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \qquad \textit{and} \qquad \mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}) \subset \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$$

of functors, which are left resp. right adjoints, are again presentable.

Definition 2.11.2. Let $\mathcal{P}\mathrm{r}^{\mathrm{L}} \subset \widehat{\mathrm{Cat}}_\infty$ denote the subcategory of presentable categories and functors, which are left adjoints.

Proposition 2.11.3. *Setting $\mathcal{C} \otimes \mathcal{D} := \mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$ for any $\mathcal{C}, \mathcal{D} \in \mathcal{P}\mathrm{r}^{\mathrm{L}}$ equips $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ with a (cartesian closed) symmetric monoidal structure*[26] *In particular, there is a functor $\mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$,*

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[26] i.e. $\otimes$ induces a cocartesian fibration $\mathcal{P}\mathrm{r}^{\mathrm{L}^\otimes} \to \mathrm{Fin}_p$, such that $\mathcal{P}\mathrm{r}^{\mathrm{L}^\otimes}_{[1]} \simeq \mathcal{P}\mathrm{r}^{\mathrm{L}}$ and $\mathcal{P}\mathrm{r}^{\mathrm{L}^\otimes}_{[n]} \simeq (\mathcal{P}\mathrm{r}^{\mathrm{L}^\otimes}_{[1]})^n$ (cf. Rmk. 2.3.4).

which preserves colimits in each variable and induces an equivalence for every $\mathcal{E} \in \mathcal{P}\mathrm{r}^{\mathrm{L}}$

$$\mathrm{Fun}^{\mathrm{bi}}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \simeq \mathrm{Fun}^{\mathrm{L}}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E})$$

where $\mathrm{Fun}^{\mathrm{bi}}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \subset \mathrm{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$ *denotes the full subcategory of functors, which preserve colimits in each variable. Moreover,* Ani *is the unit to* $\otimes$ *i.e.* $\mathcal{C} \otimes \mathrm{Ani} \simeq \mathcal{C} \simeq \mathrm{Ani} \otimes \mathcal{C}$ *for all* $\mathcal{C} \in \mathcal{P}\mathrm{r}^{\mathrm{L}}$*.*

Proposition 2.11.4. *For any (pre)-site* $(\mathcal{C}, \tau)$ *and any presentable category* $\mathcal{D}$*, there is an equivalence* $\mathrm{Shv}_\tau(\mathcal{C}) \otimes \mathcal{D} \simeq \mathrm{Shv}_\tau(\mathcal{C}; \mathcal{D})$ *and the composition*

$$L \otimes \mathrm{id}_{\mathcal{D}} \colon \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{P}(\mathcal{C})^{\mathrm{op}}, \mathcal{D}) \xrightarrow{\circ L^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{R}}(\mathrm{Shv}_\tau(\mathcal{C})^{\mathrm{op}}, \mathcal{D}) \simeq \mathrm{Shv}_\tau(\mathcal{C}; \mathcal{D})$$

exhibits $\mathrm{Shv}_\tau(\mathcal{C}; \mathcal{D})$ *as localization at the* U_τ*-local objects in the spirit of Rmk. 2.2.15.*

Lemma 2.11.5 ([SAG])**.** *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be* ∞*-topoi, then* $\mathcal{X} \otimes \mathcal{Y}$ *is an* ∞*-topos, as well. Hence,* $\mathrm{Shv}_\tau(\mathcal{C}; \mathcal{D})$ *is an* ∞*-topos, if* $\mathcal{D}$ *is.*

3 The Shape of an ∞-Topos

A fundamental strategy in algebraic topology for analysing a space X is the study of its mapping properties relative to a class of well-behaved objects. Standard homotopy theory typically focuses on maps *into* X: for instance, maps from spheres determine the homotopy groups, while maps from CW-complexes characterize the weak homotopy type $\mathrm{Sing}(X)$. For spaces with “wild” local behaviour these invariants often fail to capture relevant information. The Warsaw circle, for instance, has trivial fundamental group, but is homotopy equivalent to S^1 [Sch12; nLa25f].

To remedy this, shape theory after Mardešić and Segal [MS82], shifts the perspective to maps *out of* X into polyhedra. In this framework, a space X is approximated by a cofiltered limit in the homotopy category of polyhedra $h\mathrm{Pol}$ i.e a *pro-object* in $h\mathrm{Pol}$. Formally, this approximation of X – also known as its *polyhedral expansion* – is a universal morphism $p \colon X \to \mathbf{X}$ to a pro-object $\mathbf{X}$ in $h\mathrm{Pol}$, where X is considered as constant pro-object in $h\mathrm{Top}$ [MS82, p.24].

As opposed to the previous section, which exhibited ∞-topoi as natural setting for generalized homotopy theory, this section, develops a shape theory for ∞-topos, which directly extends the classical framework of [MS82] in case X is a compact Hausdorff space (cf. [Hoy17, 2.13], [Wan18, 4.6]). As shown in § 3.3, the pro-object approximating the ∞-topos represents its locally constant objects. To facilitate this construction, it is first necessary to develop the theory of pro-objects in the setting of ∞-categories.

3.1 Pro-Objects

An object of a category $\mathcal{C}$ is simply a functor from the one-point category into $\mathcal{C}$ i.e. the easiest functor with codomain $\mathcal{C}$. The concept of pro-objects – analogously to ind-objects – is to allow a further level of complexity and consider reasonable functors into $\mathcal{C}$, whose domain is cofiltered.

Definition 3.1.1. A **pro-object** of a small ∞-category $\mathcal{C}$ is a functor $P \colon \mathcal{C} \to \mathrm{Ani}$, which classifies a left-fibration $\mathcal{J} \to \mathcal{C}$, where $\mathcal{J}$ is cofiltered. Let $\mathrm{Pro}(\mathcal{C}) \subset \mathrm{Fun}(\mathcal{C}, \mathrm{Ani})^{\mathrm{op}}$ denote the

full subcategory of pro-objects.
As such, pro-objects are simply the opposite to ind-objects.

Lemma 3.1.2 ([HTT])**.** *For any small ∞-category $\mathcal{C}$, there is an equivalence* $\operatorname{Pro}(\mathcal{C}) \simeq \operatorname{Ind}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$.

Notation 3.1.3. As the Yoneda embedding y^{op} lands in $\operatorname{Pro}(\mathcal{C})^{\mathrm{op}}$, it is convenient to set

$$j := (y^{\mathrm{op}})^{\mathrm{op}} \colon \mathcal{C} \to \operatorname{Pro}(\mathcal{C}) \subset \operatorname{Fun}(\mathcal{C}, \mathrm{Ani})^{\mathrm{op}}.$$

Remark 3.1.4. The equivalence $\operatorname{Pro}(\mathcal{C}) \simeq \operatorname{Ind}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$ exhibits $\operatorname{Pro}(\mathcal{C})$ as the free *cofiltered* completion of $\mathcal{C}$. Indeed, every $F \in \operatorname{Ind}(\mathcal{C}^{\mathrm{op}})$ admits the following presentation via the formula for pointwise left Kan extension ([Wag23, 6.30]

$$F \simeq \operatorname{colim}\left(\mathcal{C}^{\mathrm{op}}_{/F} \xrightarrow{forget} \mathcal{C}^{\mathrm{op}} \xrightarrow{y^{\mathrm{op}}} \operatorname{Ind}(\mathcal{C}^{\mathrm{op}})\right) = \operatorname*{colim}_{(c, y^{\mathrm{op}}c \to F) \in \mathcal{C}^{\mathrm{op}}_{/F}} y^{\mathrm{op}}(c)$$

Notice, that $\mathcal{C}^{\mathrm{op}}_{/F} \simeq (\mathcal{C}_{F/})^{\mathrm{op}}$. Thus, applying $(-)^{\mathrm{op}}$ to all categories involved, exhibits $F \in \operatorname{Pro}(\mathcal{C})$ as the right Kan extension

$$F \simeq \lim\left(\mathcal{C}_{F/} \xrightarrow{forget} \mathcal{C} \xrightarrow{j} \operatorname{Pro}(\mathcal{C})\right) = \operatorname*{colim}_{(c, F \to j(c)) \in \mathcal{C}_{F/}} j(c).$$

Moreover, $\operatorname{Pro}(\mathcal{C})$ inherits a universal property from the following fact for ind-objects.

Proposition 3.1.5 ([HTT, 5.3.5.10])**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories. Suppose $\mathcal{C}$ is small and $\mathcal{D}$ admits small κ-filtered colimits for a regular cardinal κ. Then, precomposition with the Yoneda embedding induces the equivalence*

$$\operatorname{Fun}^{\kappa\text{-}\mathit{filt}}(\operatorname{Ind}_\kappa(\mathcal{C}), \mathcal{D}) \simeq \operatorname{Fun}(\mathcal{C}, \mathcal{D})$$

where $\operatorname{Fun}^{\kappa\text{-}\mathit{filt}}$ *denotes the full subcategory of* Fun *of functors, which preserve κ-filtered colimits.*

Corollary 3.1.6. *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories. Suppose $\mathcal{D}$ admits all cofiltered limits. Then,*

$$\operatorname{Fun}^{\mathrm{cofilt}}(\operatorname{Pro}(\mathcal{C}), \mathcal{D}) \xrightarrow{j^*} \operatorname{Fun}(\mathcal{C}, \mathcal{D})$$

is an equivalence, where $\operatorname{Fun}^{\mathrm{cofilt}}$ *denotes the full subcategory of* Fun *of functors, which preserve cofiltered limits.*

Proof. The previous statements yields the following equivalences:

$$\begin{aligned} \operatorname{Fun}(\mathcal{C}, \mathcal{D}) \simeq \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}^{\mathrm{op}})^{\mathrm{op}} &\overset{3.1.5}{\simeq} \operatorname{Fun}(\operatorname{Ind}(\mathcal{C}^{\mathrm{op}}), \mathcal{D}^{\mathrm{op}})^{\mathrm{op}} \simeq \operatorname{Fun}(\operatorname{Ind}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}, \mathcal{D}) \\ &\overset{3.1.2}{\simeq} \operatorname{Fun}(\operatorname{Pro}(\mathcal{C}), \mathcal{D}). \end{aligned}$$ ☐

The study of pro-objects in ∞-topoi – of which the shape is a special case – requires extending Definition 3.1.1 to large ∞-categories. To bypass the technicalities of straightening-unstraightening for large ∞-categories, the following definition is adopted.

Definition 3.1.7 ([SAG, A.8.1.1])**.** Let $\mathcal{C}$ be an accessible ∞-category with finite limits. A functor $P\colon \mathcal{C} \to \mathrm{Ani}$ is a **pro-object** of $\mathcal{C}$ if P is accessible[27] and left exact. Denote the full subcategory of pro-objects by $\mathrm{Pro}(\mathcal{C}) \subset \mathrm{Fun}(\mathcal{C}, \mathrm{Ani})^{\mathrm{op}}$.

At heart, this is the same Definition as Def.3.1.1. Indeed, [HTT, 5.3.5.4] shows, that for a small, complete ∞-category $\mathcal{C}$, a functor $P\colon \mathcal{C} \to \mathrm{Ani}$ is a pro-object in the sense of Def. 3.1.1 if and only if P is left exact. Moreover, each (general) pro-object is still a cofiltered limit of representables.

Lemma 3.1.8. *Let $\mathcal{C}$ be an accessible ∞-category with finite limits. Each $P \in \mathrm{Pro}(\mathcal{C})$ can be written as small filtered colimit in* $\mathrm{Fun}(\mathcal{C}, \mathrm{Ani})$

$$P \simeq \operatorname*{colim}_{i \in I} \mathrm{map}_{\mathcal{C}}(c_i, -)$$

where $I^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ is a cofiltered diagram. Thus, $P \simeq \lim_{i \in I} j(c_i)$ in $\mathrm{Pro}(\mathcal{C})$.
In this case, we say that $I \to \mathcal{C}$ is a presentation of P.

Proof. By the assumptions on $\mathcal{C}$ and P, there exists a regular cardinal κ, such that $\mathcal{C}$ is κ-accessible[28], P preserves κ-filtered colimits and the full subcategory of κ-compact objects $\mathcal{C}_{<\kappa} \subset \mathcal{C}$ is closed under finite limits by virtue of [HTT, 5.4.7.4]. By [HTT, 5.4.2.2], $\mathcal{C}_{<\kappa}$ is essentially small. Therefore, the restriction of P to $\mathcal{C}_{<\kappa}$ admits a description as left Kan extension

$$P|_{\mathcal{C}_{<\kappa}} \simeq \operatorname*{colim}_{c \in (\mathcal{C}^{\mathrm{op}}_{<\kappa})_{/P}} y^{\mathrm{op}}_{<\kappa}(c)$$

where $y^{\mathrm{op}}_{<\kappa}$ is the Yoneda embedding of κ-compact objects $y^{\mathrm{op}}_{<\kappa}\colon \mathcal{C}^{\mathrm{op}}_{<\kappa} \to \mathrm{Fun}(\mathcal{C}_{<\kappa}, \mathrm{Ani})$.
Recall, that $\mathcal{C}^{\mathrm{op}}_{<\kappa/P}$ is filtered. Then, define

$$P' := \mathrm{colim}_{c \in (\mathcal{C}^{\mathrm{op}}_{<\kappa})_{/P}}\, y^{\mathrm{op}}(c)$$

where y^{op} is the usual Yoneda embedding for $\mathcal{C}$. Note, that P' preserves κ-filtered colimits, since each $c \in (\mathcal{C}^{\mathrm{op}}_{<\kappa})_{/P}$ is κ-compact. Since y^{op} restricts to $y^{\mathrm{op}}_{<\kappa}$, one deduces $P|_{\mathcal{C}_{<\kappa}} \simeq P'|_{\mathcal{C}_{<\kappa}}$. This implies $P \simeq P'$, since

$$\mathrm{Fun}^{\mathrm{filt}}(\mathcal{C}, \mathrm{Ani}) \simeq \mathrm{Fun}^{\mathrm{filt}}(\mathrm{Ind}(\mathcal{C}_{<\kappa}), \mathrm{Ani}) \overset{3.1.5}{\simeq} \mathrm{Fun}(\mathcal{C}_{<\kappa}, \mathrm{Ani}). \qquad \square$$

Rmk. 3.1.4 and Prop. 3.1.5 show, that every functor $F\colon \mathcal{C} \to \mathcal{D}$ can be extended to a functor $\mathrm{Ran}_j(F)\colon \mathrm{Pro}(\mathcal{C}) \to \mathcal{D}$ via right Kan extension along j, if $\mathcal{D}$ admits small cofiltered limits. An analogue of Lem. 3.1.6 also holds for pro-objects in the sense of Def. 3.1.7.

Proposition 3.1.9 ([SAG, A.8.1.6])**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories. Suppose $\mathcal{C}$ admits finite limits and $\mathcal{D}$ admits small cofiltered limits. Then composition with the Yoneda embedding restricts to an equivalence of ∞-categories*

$$\mathrm{Fun}^{\mathrm{cofilt}}(\mathrm{Pro}(\mathcal{C}), \mathcal{D}) \longrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{D}).$$

---

[27] i.e. preserves κ-filtered colimits for some regular cardinal κ.

[28] i.e. $\mathcal{C} \simeq \mathrm{Ind}(\mathcal{C}_{<\kappa})$ for the full subcategory $\mathcal{C}_{<\kappa}$ of κ-compact objects

Proof. By Lemma 3.1.8, $\mathrm{Pro}(\mathcal{C})^{\mathrm{op}}$ is identified with the smallest subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \widehat{\mathrm{Ani}})$ that contains the essential image of the Yoneda embedding and is closed under filtered colimits. Hence, $\mathrm{Pro}(\mathcal{C})^{\mathrm{op}}$ is the free cocompletion of $\mathcal{C}^{\mathrm{op}}$ under filtered colimits (cf. Lem. 3.1.5 or [HTT, 5.3.6.2]). Consequently, $\mathrm{Pro}(\mathcal{C})^{\mathrm{op}}$ is characterized by the equivalence

$$\mathrm{Fun}^{\mathrm{filt}}(\mathrm{Pro}(\mathcal{C})^{\mathrm{op}}, \mathcal{D}^{\mathrm{op}}) \simeq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}^{\mathrm{op}}). \qquad \square$$

Remark 3.1.10. In the situation above, Lem. 3.1.8 in combination with the proof of [HTT, 5.3.6.2] show, that the extension of a functor $F \colon \mathcal{C} \to \mathcal{D}$ to $\mathrm{Pro}(\mathcal{C})$ is given by

$$\widetilde{F} \colon \mathrm{Pro}(\mathcal{C}) \to \mathcal{D},\ P \simeq \lim_{i \in I} j(c_i) \ \mapsto\ \lim_{i \in I^{\mathrm{op}}} F(c_i).$$

Definition 3.1.11 ([Car16, 2.6])**.** Precomposition with the inclusion
$i_\pi \colon \mathrm{Ani}^\pi \hookrightarrow \mathrm{Ani}$ induces a functor $i_\pi^* \colon \mathrm{Pro}(\mathrm{Ani}) \to \mathrm{Pro}(\mathrm{Ani}^\pi)$. The composition

$$\widehat{(\text{-})} : \mathrm{An} \to \mathrm{Pro}(\mathrm{An}) \xrightarrow{\tau_\pi} \mathrm{Pro}(\mathrm{An}^\pi)$$

is called **profinite completion**. Since the composition $\mathrm{An}^\pi \overset{i_\pi}{\hookrightarrow} \mathrm{An} \overset{j}{\hookrightarrow} \mathrm{Pro}(\mathrm{An})$ is accessible and preserves finite limits, i_π^* admits a right adjoint $\mathrm{Pro}(\mathrm{Ani}^\pi) \to \mathrm{Pro}(\mathrm{Ani})$. The inclusion i_p and i_{ab} of Def. 1.0.3 give rise to similar restriction maps.

Notation 3.1.12. By convenient abuse of notation one often writes $\widehat{(\text{-})}$ instead of i_π^*.

3.2 The Shape of an ∞-Topos

The *shape* of an object X of an ∞-topos $\mathcal{X}$ is a pro-object of $\mathcal{X}$ approximating X. The assignment of this approximation is fully functorial, but a pro-left adjoint even.

Definition 3.2.1 (Pro-Left Adjoint, [Hoy17])**.** Let $\mathcal{C}$ and $\mathcal{D}$ be presentable ∞-categories. Given a finite limit preserving functor $G : \mathcal{D} \to \mathcal{C}$, the composition

$$\mathcal{C}^{op} \xrightarrow{y^{\mathrm{op}}} \mathrm{Fun}(\mathcal{C}, \mathrm{Ani}) \xrightarrow{G^*} \mathrm{Fun}(\mathcal{D}, \mathrm{Ani}), \qquad c \mapsto \mathrm{map}_{\mathcal{C}}(c, G(-))$$

factors through $\mathrm{Ind}(\mathcal{D}^{op})$, since the y^{op} factors through $\mathrm{Ind}(\mathcal{C}^{\mathrm{op}})$. The resulting functor $F : \mathcal{C} \to \mathrm{Pro}(\mathcal{D})$ is called the **pro-left adjoint** to G.

Definition 3.2.2 (Shape, [Hoy17, 2.3])**.** For a given ∞-topos, the pro-left adjoint of $\underline{\Delta}_{\mathcal{X}}$ denoted by $\Pi_\infty^{\mathcal{X}} : \mathcal{X} \to \mathrm{Pro}(\mathrm{Ani})$ is called the **shape** associated to $\mathcal{X}$.
Hence, for any $X \in \mathcal{X}$, the shape of X is the pro-anima

$$\Pi_\infty^{\mathcal{X}}(X) := \mathrm{map}_{\mathcal{X}}(X, \underline{\Delta}(-)) : \mathrm{Ani} \to \mathrm{Ani}, \qquad K \mapsto \mathrm{map}_{\mathcal{X}}(X, \underline{K})$$

Moreover, define $\Pi_\infty \mathcal{X} := \Pi_\infty^{\mathcal{X}}(1_{\mathcal{X}})$ and call it the **shape of the (whole) ∞-topos $\mathcal{X}$**.

Although this terminology slightly overloads natural language, a clear distinction in notation should prevent ambiguity: $\Pi_\infty^{\mathcal{X}} : \mathcal{X} \to \mathrm{Pro}(\mathrm{Ani})$ denotes the *diagram* of pro-anima associated to $\mathcal{X}$, while $\Pi_\infty \mathcal{X} \in \mathrm{Pro}(\mathrm{Ani})$ denotes the resulting *explicit pro-anima.*

Notation 3.2.3. Let p be a prime and $\mathcal{X}$ an ∞-topos. Def 1.0.3 yields restrictions of $\Pi^{\mathcal{X}}_\infty$:

$$\begin{aligned}
\widehat{\Pi}^{\mathcal{X}}_\infty &\colon \mathcal{X} \xrightarrow{\Pi^{\mathcal{X}}_\infty} \mathrm{Pro}(\mathrm{Ani}) \xrightarrow{i^*_\pi} \mathrm{Pro}(\mathrm{Ani}^\pi) \\
{}^p\widehat{\Pi}^{\mathcal{X}}_\infty &\colon \mathcal{X} \xrightarrow{\Pi^{\mathcal{X}}_\infty} \mathrm{Pro}(\mathrm{Ani}) \xrightarrow{i^*_p} \mathrm{Pro}(\mathrm{Ani}^{p\text{-free}}) \\
{}^{\mathrm{ab}}\Pi^{\mathcal{X}}_\infty &\colon \mathcal{X} \xrightarrow{\Pi^{\mathcal{X}}_\infty} \mathrm{Pro}(\mathrm{Ani}) \xrightarrow{i^*_{\mathrm{ab}}} \mathrm{Pro}(\mathrm{Ani}^{\mathrm{ab}}) \\
{}^{\mathrm{ab}}\widehat{\Pi}^{\mathcal{X}}_\infty &\colon \mathcal{X} \xrightarrow{\Pi^{\mathcal{X}}_\infty} \mathrm{Pro}(\mathrm{Ani}) \xrightarrow{i^*_{\pi,\mathrm{ab}}} \mathrm{Pro}(\mathrm{Ani}^\pi \cap \mathrm{Ani}^{\mathrm{ab}}).
\end{aligned}$$

While [HTT, 7.1.6.6] suggests that the assignment $\mathcal{X} \mapsto \Pi_\infty \mathcal{X}$ is functorial in $\mathcal{RT}\mathrm{op}$ an explicit verification seems to absent form the literature. A formal proof is provided below.

Lemma 3.2.4. *The assignment* $\mathcal{X} \mapsto \Pi_\infty \mathcal{X}$ *yields a functor* $\Pi_\infty : \mathcal{RT}\mathrm{op} \to \mathrm{Pro}(\mathrm{Ani})$.

Proof. Consider the composition of functors

$$\phi : \mathcal{RT}\mathrm{op}^{op} \times \mathrm{Ani} \xrightarrow{\mathrm{id}\times \mathrm{Ani}_{/-}} \mathcal{RT}\mathrm{op}^{op} \times \mathcal{RT}\mathrm{op} \xrightarrow{\mathrm{map}_{\mathcal{RT}\mathrm{op}}(-,-)} \mathrm{Ani}$$

Notice, that $\phi(\mathcal{X}, K) = \mathrm{map}_{\mathcal{RT}\mathrm{op}}(\mathcal{X}, \mathrm{Ani}_{/K}) \simeq \Pi_\infty(\mathcal{X})(K)$ by Lemma 2.4.10 for every $\mathcal{X} \in \mathcal{RT}\mathrm{op}$ and $K \in \mathrm{Ani}$. Let

$$\psi \colon \mathcal{RT}\mathrm{op}^{\mathrm{op}} \to \mathrm{Fun}(\mathrm{Ani}, \mathrm{Ani})$$

be the functor corresponding to ϕ under the $\times \dashv \mathrm{Fun}$ adjunction. Then, $\Pi_\infty \simeq \psi^{\mathrm{op}}$. ☐

At first glance, the Definition of the shape of an ∞-topos varies throughout the literature. Although, as verified in the next example, all common notions can be recovered from Def. 3.2.2.

Example 3.2.5. In [HTT], Lurie defines the shape of an ∞-topos $\mathcal{X}$ as the composition

$$\Pi^{Lurie}_\infty(\mathcal{X}) : \mathrm{Ani} \overset{\underline{\Delta}}{\to} \mathcal{X} \overset{\Gamma}{\to} \mathrm{Ani}$$

Lurie's definition coincides with Π_∞ from Def. 3.2.2 above, since

$$\begin{aligned}
\Pi^{Lurie}_\infty(\mathcal{X}) &= \Gamma \circ \underline{\Delta} \simeq \mathrm{map}_{\mathrm{Ani}}(*, \Gamma \circ \underline{\Delta}(-)) \\
&\simeq \mathrm{map}_{\mathcal{X}}(\underline{\Delta}(*), \underline{\Delta}(-)) \simeq \mathrm{map}_{\mathcal{X}}(1_{\mathcal{X}}, \underline{\Delta}(-)) = \Pi_\infty \mathcal{X}.
\end{aligned}$$

Enhancing Π^{Lurie}_∞, Carchedi [Car16] defines the shape functor[29] associated to $\mathcal{X}$ by

$$\Pi^{Car}_\infty : \mathcal{X} \overset{(\mathcal{X}_{/-})_*}{\to} \mathcal{RT}\mathrm{op} \overset{\Pi_\infty}{\to} \mathrm{Pro}(\mathrm{Ani})$$

where $(\mathcal{X}_{/-})_*$ is the functor of This functor also coincides with the shape functor $\Pi^{\mathcal{X}}_\infty$ of Def. 3.2.2. To verify this, observe, that for every $X \in \mathcal{X}$ with unique morphism $t \colon X \to 1_{\mathcal{X}}$ there are

---

[29] Carchedi [Car16], however, does not seem show that Π_∞ is a functor.

equivalence

$$\begin{aligned}\Pi^{\mathrm{car}}_{\infty}\mathcal{X} &= \Gamma_{\mathcal{X}_{/X}}\,\underline{\Delta}_{\mathcal{X}_{/X}} \simeq \Gamma_{\mathcal{X}} t_* t^*\,\underline{\Delta}_{\mathcal{X}} \simeq \mathrm{map}_{\mathcal{X}}(1_{\mathcal{X}}, t_*t^*\,\underline{\Delta}_{\mathcal{X}}(-)) \simeq \mathrm{map}_{\mathcal{X}_{/X}}(t^*1_{\mathcal{X}}, t^*\,\underline{\Delta}_{\mathcal{X}}(-))\\ &\simeq \mathrm{map}_{\mathcal{X}_{/X}}(1_{\mathcal{X}_{/X}}, t^*\,\underline{\Delta}_{\mathcal{X}}(-)) \overset{\mathrm{A.1.3}}{\simeq} \mathrm{map}_{\mathcal{X}}(X, \underline{\Delta}_{\mathcal{X}}(-)).\end{aligned}$$

The first equivalence persists, since Ani is terminal in $\mathcal{RT}$op, resp. initial in $\mathcal{LT}$op.

Even the assignment $\mathcal{X} \mapsto \Pi^{\mathcal{X}}_{\infty} \in \mathrm{Fun}(\mathcal{X}, \mathrm{Pro}(\mathrm{Ani}))$ is “1-natural” in the following sense.

Corollary 3.2.6. *For any geometric morphism* $\mathcal{X} \underset{g_*}{\overset{g^*}{\leftrightarrows}} \mathcal{Y}$*, there is a morphism in* $\mathrm{Fun}(\mathcal{X}, \mathrm{Pro}(\mathrm{Ani}))$

$$\alpha_{g_*} : \Pi^{\mathcal{X}}_{\infty} \Rightarrow \Pi^{\mathcal{Y}}_{\infty} \circ g_*$$

obtained by composing the natural transformation $(\mathcal{X}_{/-})_* \Rightarrow (\mathcal{Y}_{/-})_* \circ g_*$ *with* Π_{∞}*. Likewise, one obtains a morphism in* $\mathrm{Fun}(\mathcal{Y}, \mathrm{Pro}(\mathrm{Ani}))$

$$\alpha_{g^*} : \Pi^{\mathcal{X}}_{\infty} \circ g^* \Rightarrow \Pi^{\mathcal{Y}}_{\infty}.$$

by postcomposing $\mathcal{Y}^*_{/-} \Rightarrow \mathcal{X}^*_{/-} \circ (g^*)^{\mathrm{op}}$ *with* $(\Pi_{\infty})^{\mathrm{op}} \colon \mathcal{LT}\mathrm{op} \to \mathrm{Fun}(\mathrm{Ani}, \mathrm{Ani})$*.*
Here, $\mathcal{Y}^*_{/-} \in \mathrm{Fun}(\mathcal{Y}^{\mathrm{op}}, \mathcal{LT}\mathrm{op})$*,* $\mathcal{X}^*_{/-} \in \mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathcal{LT}\mathrm{op})$ *and their analogues* $(\mathcal{Y}_{/-})_*$ *and* $(\mathcal{X}_{/-})_*$ *are the functors of Cor. 2.4.5.*

Remark 3.2.7. Under the assumption that slicing is a fully natural operation (cf. Rmk A.1.11), the functor $\Pi_{\infty} \colon \mathcal{X} \mapsto \Pi^{\mathcal{X}}_{\infty}$ is fully natural as well i.e. extends to a functor with domain $\mathcal{RT}$op.

Lemma 3.2.8. *For any* ∞*-topos* $\mathcal{X}$*, let* $\Pi^{<\infty}_{\infty}\mathcal{X} := i^*_{<\infty}(\Pi_{\infty}\,\mathcal{X})$ *denote the restriction of* $\Pi_{\infty}\,\mathcal{X}$ *to* $\mathrm{Pro}(\mathrm{Ani}^{<\infty})$*. Hypercompletion induces an equivalence*

$$\Pi^{<\infty}_{\infty}\mathcal{X}^{\mathrm{hyp}} \overset{\simeq}{\Longrightarrow} \Pi^{<\infty}_{\infty}\mathcal{X}.$$

In particular, hypercompletion produces the equivalences

$$i^*_{<\infty} \circ \Pi^{\mathcal{X}^{\mathrm{hyp}}}_{\infty} \circ \mathfrak{h}\mathrm{yp} \overset{\simeq}{\Longrightarrow} i^*_{<\infty} \circ \Pi^{\mathcal{X}}_{\infty}, \qquad \widehat{\Pi}^{\mathcal{X}^{\mathrm{hyp}}}_{\infty} \circ \mathfrak{h}\mathrm{yp} \overset{\simeq}{\Longrightarrow} \widehat{\Pi}^{\mathcal{X}}_{\infty}, \qquad {}^{p}\widehat{\Pi}^{\mathcal{X}^{\mathrm{hyp}}}_{\infty} \circ \mathfrak{h}\mathrm{yp} \overset{\simeq}{\Longrightarrow} {}^{p}\widehat{\Pi}^{\mathcal{X}}_{\infty}.$$

Proof. Let u denote the fully faithful right adjoint to $\mathfrak{h}$yp. Since $\underline{\Delta}_{\mathcal{X}}(A) \in \mathcal{X}$ is truncated, whenever $A \in \mathrm{Ani}^{<\infty}$ (cf.Prop 2.6.4) and truncated objects are hypercomplete (cf.Prop. 2.10.5), there are equivalences for every $A \in \mathrm{Ani}^{<\infty}$:

$$\begin{aligned}\Pi^{<\infty}_{\infty}\mathcal{X}^{\mathrm{hyp}}(A) = \mathrm{map}_{\mathcal{X}^{\mathrm{hyp}}}(1_{\mathcal{X}^{\mathrm{hyp}}}, \underline{\Delta}_{\mathcal{X}^{\mathrm{hyp}}}\,A) &\overset{u(1_{\mathcal{X}^{\mathrm{hyp}}})\simeq 1_{\mathcal{X}}}{\simeq} \mathrm{map}_{\mathcal{X}}(1_{\mathcal{X}}, u\,\underline{\Delta}_{\mathcal{X}^{\mathrm{hyp}}}\,A)\\ \overset{\underline{\Delta}_{\mathcal{X}^{\mathrm{hyp}}}\simeq \mathfrak{h}\mathrm{yp}\,\underline{\Delta}_{\mathcal{X}}}{\simeq} \mathrm{map}_{\mathcal{X}}(1_{\mathcal{X}}, \underline{\Delta}_{\mathcal{X}}(A)) &= \Pi^{<\infty}_{\infty}\mathcal{X}(A).\end{aligned}$$

As this equivalence holds for every ∞-topos, it shows that the natural transformations in the second assertion are pointwise equivalences, since for every $X \in \mathcal{X}$, we have

$$i^*_{<\infty} \circ \Pi^{\mathcal{X}^{\mathrm{hyp}}}_{\infty}(\mathfrak{h}\mathrm{yp}X) \overset{3.2.5}{\simeq} \Pi^{<\infty}_{\infty}\mathcal{X}^{\mathrm{hyp}}_{/X} \simeq \Pi^{<\infty}_{\infty}\mathcal{X}_{/X} \overset{3.2.5}{\simeq} i^*_{<\infty} \circ \Pi^{\mathcal{X}}_{\infty}(X). \qquad \square$$

The other assertions follow instantly since $\mathrm{Ani}^{p\text{-free}} \subset \mathrm{Ani}^{\pi} \subset \mathrm{Ani}^{<\infty}$.

This motivates the following definition.

Definition 3.2.9. An ∞-topos $\mathcal{X}$ is said to be **locally n-connected** for a $0 \leq n < \infty$ if the n-truncation $\tau_{\leq n}\,\underline{\Delta}_{\mathcal{X}}\colon \tau_{\leq n}\mathrm{Ani} \to \tau_{\leq n}\mathcal{X}$ admits a left adjoint $\Pi_n^{\mathcal{X}}$. This is extended to case $n = \infty$ by referring to $\mathcal{X}$ as **locally ∞-connected**, if $\underline{\Delta}$ itself admits a left adjoint $\Pi_{\mathrm{const}}^{\mathcal{X}}$ In this case, the anima $\Pi_{\mathrm{const}}^{\mathcal{X}}(1_{\mathcal{X}})$ is the **constant shape of $\mathcal{X}$**.

Notation 3.2.10. For a terminal object $1_{\mathcal{X}}$, $\Pi\mathcal{X}_n 1_{\mathcal{X}}$ is usually abbreviated by $\Pi_n\mathcal{X}$.

Observation 3.2.11. Let $\mathcal{X}$ be a locally ∞-connected ∞-topos. Then, $\Pi_\infty^{\mathcal{X}} \simeq j \circ \Pi_{\mathrm{const}}$ since the adjunction $\Pi_{\mathrm{const}} \dashv \underline{\Delta}_{\mathcal{X}}$ provides the following equivalence for every $X \in \mathcal{X}$:

$$j \circ \Pi_{\mathrm{const}}(X)\,(-) = \mathrm{map}_{\mathrm{An}}(\Pi_{\mathrm{const}}(X), -) \simeq \mathrm{map}_{\mathcal{X}}(X, \underline{\Delta}_{\mathcal{X}}(-)) = \Pi_\infty^{\mathcal{X}}(X).$$

Observation 3.2.12. Every presheaf-∞-topos is locally ∞-connected, since for any small ∞-category $\mathcal{C}$ the functor $\underline{\Delta}\colon \mathrm{Ani} \to \mathcal{P}(\mathcal{C})$ is right adjoint to the colimit functor

$$\operatorname{colim}\colon \mathcal{P}(\mathcal{C}) \to \mathrm{Ani}, \quad (P\colon \mathcal{C} \to \mathrm{Ani}) \mapsto \operatorname{colim}_{\mathcal{C}} P.$$

This is because $\underline{A} \in \mathcal{P}(\mathcal{C})$, is the functor $\mathcal{C}^{\mathrm{op}}\colon \mathcal{C}^{\mathrm{op}} \to * \xrightarrow{A} \mathrm{Ani}$ by Lem. 2.3.25 and thus $\mathrm{map}(P, \underline{A}) \simeq \mathrm{map}(\operatorname{colim}_{\mathcal{C}} P, A)$ by the usual colimit $\dashv$ const adjunction (see [nLa26a]).

Observation 3.2.13. If $\mathcal{X}$ is locally ∞-connected, then for every $X \in \mathcal{X}$, $\mathcal{X}_{/X}$ is locally ∞-connected as well. This is verified by considering the unique map $X \to 1_{\mathcal{X}}$ to a terminal object and the induced the composition of essential geometric morphisms

$$\begin{tikzcd}
\mathrm{Ani} \arrow[r, "\underline{\Delta}_{\mathcal{X}}"] & \mathcal{X} \arrow[l, bend right=50, "\Pi_{\mathrm{const}}^{\mathcal{X}}"'] \arrow[l, bend left=50, "\Gamma_{\mathcal{X}}"] \arrow[r, "t^*"] & \mathcal{X}_{/X} \arrow[l, bend right=50, "t_!"'] \arrow[l, bend left=50, "t_*"]
\end{tikzcd}$$

Let us record a general procedure to detect ∞-topoi with constant shape. The following Definition is a direct generalization of the usual 1-categorical definition to the world of ∞-categories.

Definition 3.2.14 ([nLa26b, 2.1])**.** A ∞-site $(\mathcal{C}, \tau)$ is called locally contractible if every constant presheaf on $\mathcal{C}$ is a sheaf for the topology τ.

Proposition 3.2.15 ([nLa26b, 3.1])**.** *If (C, τ) be a locally contractible ∞-site, then $\mathrm{Shv}_\tau(C)$ is locally ∞-connected.*

Proof. Consider the composition $\operatorname{colim}_{\mathcal{C}} \circ i\colon \mathrm{Shv}_\tau(\mathcal{C}) \to \mathcal{P}(\mathcal{C}) \to \mathrm{Ani}$. This functor provides is left adjoint to $\underline{\Delta}_{\mathrm{Shv}_\tau(\mathcal{C})}$, since for every $X \in \mathrm{Shv}_\tau(\mathcal{C})$ and $A \in \mathrm{Ani}$

$$\mathrm{map}_{\mathrm{Shv}_\tau(\mathcal{C})}(X, \underline{\Delta}_{\mathrm{Shv}_\tau(\mathcal{C})}(A)) \qquad \mathrm{map} \qquad (iX, iL_\tau\,\underline{\Delta}_{\mathcal{P}(\mathcal{C})}(A))$$
$$\text{i fully faithful} \overset{2.3.25}{\simeq} {}_{\mathcal{P}(\mathcal{C})}$$
$$\overset{\text{locally contractible}}{\simeq} \mathrm{map}_{\mathcal{P}(\mathcal{C})}(iX, \underline{\Delta}_{\mathcal{P}(\mathcal{C})}(A)) \simeq \mathrm{map}_{\mathrm{Ani}}(\operatorname{colim}_{\mathcal{C}} iX, A) \qquad \square$$

Lemma 3.2.16. *Let $(\mathcal{C})$ be a small ∞-category. If $\mathcal{X}$ is an ∞-topos and $L\colon \mathcal{P}(\mathcal{C}) \to \mathcal{X}$ a* *localization, such that* $\mathcal{X}_{/Ly(c)}$ *is locally* ∞*-connected for each* $c \in C$*, then* $\mathcal{X}$ *is locally* ∞*-connected as well.*

Proof. Since every presheaf is a colimit of representables, we may define a functor $\Pi^{\mathcal{X}}_{\mathrm{const}} \colon \mathcal{X} \to \mathrm{Ani}$ by setting

$$\Pi^{\mathcal{X}}_{\mathrm{const}}(X) := \mathrm{colim}_{i \in I}\, \Pi^{\mathcal{X}_{/Ly(c)}}_{\mathrm{const}}(1_{\mathcal{X}_{/Ly(c_i)}})$$

for every $X \in \mathcal{X}$, which is given by $X \simeq \mathrm{colim}_{i \in I}\, Ly(c_i)$. Then, for every such $X \in \mathcal{X}$ and any $A \in \mathrm{Ani}$, we have

$$\begin{aligned}\mathrm{map}_{\mathcal{X}}(X, \underline{\Delta}_{\mathcal{X}}(A)) &\simeq \lim_{i \in I} \mathrm{map}(Ly(c_i), \underline{\Delta}_{\mathcal{X}}(A)) \simeq \lim_{i \in I} \mathrm{map}_{\mathcal{X}_{/Ly(c_i)}}(1_{\mathcal{X}_{/Ly(c_i)}}, \underline{\Delta}_{\mathcal{X}_{/Ly(c_i)}}(A)) \\ &\simeq \lim_{i \in I} \mathrm{map}_{\mathrm{Ani}}(\Pi^{\mathcal{X}_{/Ly(c_i)}}_{\mathrm{const}}(1_{\mathcal{X}_{/Ly(c_i)}}, A) \simeq \mathrm{map}_{\mathrm{Ani}}(\Pi^{\mathcal{X}}_{\mathrm{const}}(X), A).\end{aligned}$$ ☐

Given the generality of the procedure of this proof, one should be able to retrieve $\Pi^{\mathcal{X}}_{\mathrm{const}}$ as some form of left Kan extension.

Corollary 3.2.17. *Let* X *be a locally contractible topological space, then* $\mathrm{Shv}^{hyp}_{\mathrm{open}}(X)$ *is locally* ∞*-connected. In particular,* $\widehat{\Pi}^{\mathrm{Shv}_{\mathrm{open}}(X)}_{\infty} \simeq j \circ \Pi^{\mathrm{Shv}_{\mathrm{open}}(X)}_{\mathrm{const}}$ *by*

Proof. Let $c\,\mathrm{Op}(X) \subset \mathrm{Op}(X)$ denote the subsite of contractible open subsets of X. It can be shown that $c\,\mathrm{Op}(X)$ forms a contractible ∞-site. See [nLa26b, 4.1] for details. ☐

Remark 3.2.18. As outlined in [HA, A.4] and [nLa26c, 3.6], the shape of the ∞-topos of a locally contractible space X coincides with the singular set

$$\Pi^{\mathrm{Shv}_{\mathrm{open}}(X)}_{\mathrm{const}}(1_X) \simeq \mathrm{Sing}(X), \qquad \text{thus} \qquad \Pi_\infty \mathrm{Shv}_{\mathrm{open}}(X)^{\mathrm{hyp}} \simeq j(\mathrm{Sing}(X)).$$

Corollary 3.2.19. *Let* $\mathrm{Top}^{\mathrm{lContr}} \subset \mathrm{Top}$ *denote the full subcategory of locally contractible spaces. Then,* $\mathrm{Shv}^{\mathrm{hyp}}_{\mathrm{open}}(\mathrm{Top}^{\mathrm{lContr}})$ *is locally* ∞*-connected.*

3.3 Representation of Locally Constant ∞-Sheaves

This section establishes the definition of locally constant objects of an ∞-topos $\mathcal{X}$ and related notions, all deduced from the various finiteness conditions for anima introduced in Def. 1.0.3. Following [Hoy17; BH20], $\Pi_\infty \mathcal{X}$ provides a description of locally constant objects in terms of *local systems*, depending on the local connectivity of $\mathcal{X}$. Since the accounts in [Hoy17; BH20] are primarily sketches, this section provides a detailed treatment of the arguments leading to this result.

Definition 3.3.1. . Let $\mathcal{X}$ be an ∞-topos. An object $X \in \mathcal{X}$ is **locally constant**, if it satisfies the condition

$\boxed{\Delta}$**:** There is an effective epimorphism $\coprod_{i \in I} U_\alpha \to 1_{\mathcal{X}}$ and for each $i \in I$ there is an $K_i \in \mathrm{Ani}$ such that$X \times U_i \simeq \underline{K_i} \times U_i$ in $\mathcal{X}_{/U_i}$.

Furthermore, an $X \in \mathcal{X}$ is **finite locally constant**, if it is locally constant and all K_i appearing condition $\boxed{\Delta}$ in are π-finite.

Specialising these notions, a (finite) locally constant object $X \in \mathcal{X}$ **abelian**, if all K_i appearing in condition $\boxed{\Delta}$ are in particular abelian.

Moreover, an object $X \in \mathcal{X}$ is **(finite) locally constant constructible** or **lissé**, if it is (finite) locally constant and the indexing set I can be chosen to be finite.

Lastly, a finite locally constant $X \in \mathcal{X}$ has **p-coprime stalks**, or is **p-coprime** for short, if all K_i in condition $\boxed{\Delta}$ have p-coprime homotopy groups. Denote by

- $\mathcal{X}^{\mathrm{lc}}$ the full subcategory spanned by locally constant objects.
- $\mathcal{X}^{\mathrm{flc}} \subset \mathcal{X}^{\mathrm{lc}}$ the full subcategory of finite locally constant objects.
- $\mathcal{X}^{\mathrm{lcc}} \subset \mathcal{X}^{\mathrm{lc}}$ the full subcategory of locally constant constructible objects.
- $\mathcal{X}^{\mathrm{lissé}} \subset \mathcal{X}^{\mathrm{lcc}}$ the full subcategory of finite locally constant constructible i.e. lissé objects.

Observation 3.3.2. Morphisms in $\mathcal{LT}\mathrm{op}$ carry locally constant objects to locally constant objects, since they preserve effective epimorphisms, coproducts and constant objects.

The newly introduced notions specialise coherent objects for locally coherent ∞-topoi.

Proposition 3.3.3 ([DAGXIII, Lem.2.3.21])**.** *For any locally coherent ∞-topos $\mathcal{X}$, every finite locally constant object is truncated coherent, i.e. there is an inclusion $\mathcal{X}^{\mathrm{flc}} \subset \mathcal{X}^{\mathrm{lissé}}$.*

Proof. Omitted. For $\mathcal{X} = \mathrm{Shv}_{\acute{e}t}(X)$ for a qcqs scheme or analytic space X follows from 7.1.16. ☐

Before coming to the description of locally constant objects via Π_∞, let us record a general procedure to reduce equivalences of global sections of locally constant sheaves to constant sheaves.

Lemma 3.3.4. *Let $f^*\colon \mathcal{X} \to \mathcal{Y}$ be a geometric morphism and let $X \in \mathcal{X}$ be locally constant. By Definition, X comes with an effective epimorphism $\coprod_{i \in I} U_i \to 1_{\mathcal{X}}$ and a family $\{K_i\}_{i \in I} \subset \mathrm{Ani}$, such that $U_i \times X \simeq U_i \times \underline{K_i}$. Suppose f^* induces equivalences $\Gamma_{\mathcal{X}_{/U}}(\underline{K}) \simeq \Gamma_{\mathcal{Y}_{/f^*(U)}}(\underline{K})$ for all $U \in \mathcal{X}$ and $K \in \mathrm{Ani}$. Then, f^* also induces an equivalence $\Gamma_{\mathcal{X}}(X) \simeq \Gamma_{\mathcal{Y}}(f^*(X))$.*

Proof. Since colimits commute with colimits and map is commutes with colimits, there are the following equivalences

$$\begin{aligned}
&\Gamma_{\mathcal{X}}(X) = \mathrm{map}_{\mathcal{X}}(1_{\mathcal{X}},\, X) \simeq \mathrm{map}_{\mathcal{X}}(\operatorname*{colim}_{\mathrm{N}(\Delta^{\mathrm{op}})} \check{\mathrm{C}}(\coprod_{i\in I} U_i \to 1_{\mathcal{X}}),\, X)\\
&\simeq \lim_{[n]\in\mathrm{N}(\Delta^{\mathrm{op}})} \prod_{\underline{i}=(i_1,\dots,i_n)\in I^n} \mathrm{map}_{\mathcal{X}}(U_{i_1}\times\cdots\times U_{i_n},\, X)\\
&\overset{A.1.3}{\simeq} \lim_{\mathrm{N}(\Delta^{\mathrm{op}})} \prod_{\underline{i}\in I^n} \mathrm{map}_{\mathcal{X}_{U_{i_1}\times\cdots\times U_{i_n}}}(1_{U_{i_1}\times\dots U_{i_n}},\, X\times U_{i_1}\times\cdots\times U_{i_n})\\
&\simeq \lim_{\mathrm{N}(\Delta^{\mathrm{op}})} \prod_{\underline{i}\in I^n} \mathrm{map}_{\mathcal{X}_{U_{i_1}\times\cdots\times U_{i_n}}}(1_{U_{i_1}\times\dots U_{i_n}},\, K_{i_1}\times U_{i_1}\times\cdots\times U_{i_n})\\
&= \lim_{\mathrm{N}(\Delta^{\mathrm{op}})} \prod_{\underline{i}\in I^n} \Gamma_{\mathcal{X}_{/U_{i_1}\times\dots U_{i_n}}}(\underline{K_{i_1}}) \simeq \lim_{\mathrm{N}(\Delta^{\mathrm{op}})} \prod_{\underline{i}\in I^n} \Gamma_{\mathcal{Y}_{/f^*(U_{i_1}\times\dots U_{i_n})}}(\underline{K_{i_1}})\\
&\simeq \Gamma_{\mathcal{Y}}(f^*X).
\end{aligned}$$

□

To obtain a representation of locally constant sheaves by Π_∞, we begin by observing that $\Pi_\infty^{\mathcal{X}}$ – the pro-left adjoint to $\underline{\Delta}$ – naturally admits a actual right adjoint.

Observation 3.3.5. Consider the slicing functor $(\mathrm{Ani}_{/-})_*\colon \mathrm{Ani} \to \mathcal{RT}\mathrm{op}^{\mathrm{ess}} \subset \mathcal{RT}\mathrm{op}$. By Prop. 2.1.5 $\mathcal{RT}\mathrm{op}$ admits small cofiltered limits. Thus, Cor. 3.1.6 implies that $(\mathrm{Ani}_{/-})_*$ extends to a functor

$$\varrho(-)\colon\ \mathrm{Pro}(\mathrm{Ani}) \to \mathcal{RT}\mathrm{op}^{\mathrm{ess}} \subset \mathcal{RT}\mathrm{op}.$$

In case $\mathcal{X}$ is a locally ∞-connected ∞-topos, this functor is simply given by

$$\varrho(\Pi_\infty\,\mathcal{X}) \simeq \varrho(j(\Pi_{\mathrm{const}}(1_{\mathcal{X}}))) \overset{\text{Obs. 3.2.11}}{\simeq} \mathrm{Ani}/\Pi_{\mathrm{const}}(1_{\mathcal{X}}).$$

Lemma 3.3.6. *The functor ϱ is right adjoint to Π_∞ i.e.*

$$(\Pi_\infty \vdash \varrho(-))\colon\ \mathrm{Pro}(\mathrm{Ani}) \underset{\varrho(-)}{\overset{\Pi_\infty}{\leftrightarrows}} \mathcal{RT}\mathrm{op}$$

Thus, the unit of this adjunction induces a geometric morphism $\varphi_*\colon \mathcal{X} \to \varrho(\Pi_\infty\,\mathcal{X})$.

Note. We intentionally denote this functor by 'ϱ', with the mnemonic in mind: '**rho** sounds like '**pro**', so it must be the right adjoint on **pro**-anima.

Proof. Since ϱ is obtained from $(\mathrm{Ani}_{/-})_*$ by right Kan extension i.e. by setting

$$\varrho(P) := \lim_{(K,P\to j(K))\,\in {}^{P}\!/_{\mathrm{Ani}}} {}^{\mathrm{Ani}}\!/_{K}$$

for every $P \in \mathrm{Pro}(\mathrm{Ani})$. Now for every $P \in \mathrm{Pro}(\mathrm{Ani})$ and $\mathcal{X} \in \mathcal{RT}\mathrm{op}$, we have

$$\begin{aligned}
\mathrm{map}_{\mathcal{RT}\mathrm{op}}(\mathcal{X}, \varrho(P)) \quad &\simeq \lim_{(K,P\to j(K))\,\in {}^{P}\!/_{\mathrm{Ani}}} \mathrm{map}_{\mathcal{RT}\mathrm{op}}(\mathcal{X}, {}^{\mathrm{Ani}}\!/_{K})\\
\overset{\text{Cor 2.4.10}}{\simeq} \mathrm{map}_{\mathcal{X}}(1_{\mathcal{X}}, \underline{\Delta}_{\mathcal{X}}(K)) \quad &= \lim_{(K,P\to j(K))\,\in {}^{P}\!/_{\mathrm{Ani}}} \Pi_\infty\,\mathcal{X}(K)\\
\simeq \lim_{(K,P\to j(K))\,\in {}^{P}\!/_{\mathrm{Ani}}} \mathrm{map}_{\mathrm{Pro}(\mathrm{Ani})}(\Pi_\infty\,\mathcal{X}, j(K)) &\simeq \quad \mathrm{map}_{\mathrm{Pro}(\mathrm{Ani})}(\Pi_\infty\,\mathcal{X}, P)
\end{aligned}$$

$\square$

Observation 3.3.7. Composing this adjunctions with the (abelian) profinite yields refinements of ψ^*. Indeed consider the compositions

$$\mathrm{Pro}(\mathrm{Ani}^{\pi}) \underset{\widehat{(-)}}{\overset{i_\pi}{\leftrightarrows}} \mathrm{Pro}(\mathrm{Ani}) \underset{\Pi_\infty}{\overset{\varrho}{\leftrightarrows}} \mathcal{RT}\mathrm{op}\, \mathrm{Pro}(\mathrm{Ani}^{\pi,\mathrm{ab}}) \underset{\widehat{(-)}^{\mathrm{ab}}}{\overset{i_\pi}{\leftrightarrows}} \mathrm{Pro}(\mathrm{Ani}) \underset{\Pi_\infty}{\overset{\varrho}{\leftrightarrows}} \mathcal{RT}\mathrm{op}$$

The composed functors are again adjoint (cf. [Lan21, 5.1.5]). Therefore, the composed units produce natural geometric morphism for every ∞-topos $\mathcal{X}$:

$$\psi^*\colon \varrho(i_\pi\, \widehat{\Pi}_\infty \mathcal{X}) \to \varrho(\Pi_\infty \mathcal{X}) \xrightarrow{\varphi^*} \mathcal{X} \quad \text{and} \quad {}^{\mathrm{ab}}\psi^*\colon \varrho(i_\pi\, \widehat{\Pi}^{\mathrm{ab}}_\infty \mathcal{X}) \to \varrho(\Pi_\infty \mathcal{X}) \xrightarrow{\varphi^*} \mathcal{X}$$

Note that there are two more equivalent ways to view the functor ϱ.

Remark 3.3.8. Considering the slicing functor $\mathrm{Ani}^*_{/-}\colon \mathrm{Ani}^{\mathrm{op}} \to \mathcal{LT}\mathrm{op}$ instead of $(\mathrm{Ani}_{/-})_*$, Cor. 3.1.6 yields the functor

$$\varrho^*\colon\ \mathrm{Pro}(\mathrm{Ani})^{\mathrm{op}} \to \mathcal{LT}\mathrm{op}, \quad (I \xrightarrow{P} \mathrm{Ani}) \mapsto (I^{\mathrm{op}} \xrightarrow{P^{\mathrm{op}}} \mathrm{Ani}^{\mathrm{op}} \xrightarrow{\mathrm{Ani}_{/-}} \mathcal{LT}\mathrm{op})$$

which is anti-equivalent to ϱ, as a consequence of Prop. 2.1.4. Here we view $\mathrm{Pro}(\mathrm{Ani})^{\mathrm{op}}$ as filtered diagrams in Ani via Lem. 3.1.8.

For the second viewpoint, it is convenient to "extend" functor categories to pro-object.

Notation 3.3.9. Let $P \in \mathrm{Pro}(\mathcal{C})$ be an pro-object in an ∞-category $\mathcal{C}$, presented as $P \simeq \lim_{i\in I} j(K_i)$ for some cofiltered diagram $K\colon I \to \mathcal{C}$. Given a ∞-category $\mathcal{D}$, set

$$\mathrm{Fun}(P, \mathcal{D}) := \operatorname*{colim}_{i \in I^{\mathrm{op}}} \mathrm{Fun}(K_i, \mathcal{D})$$

where the filtered colimit is taken in $\widehat{\mathrm{Cat}}_\infty$.

In these terms $\mathrm{Fun}(-, \mathrm{Ani})$ models the straightening of ϱ^*.

Remark 3.3.10. Let $f\colon K \to L$ be a map of anima. Under the straightening equivalence [Lan21, 3.3.17] the associated adjoint triple

$$\begin{tikzcd}
\mathrm{Ani}_{/L} \arrow[r, "f^*"] & \mathrm{Ani}_{/K} \arrow[l, bend right=40, "f_!"'] \arrow[l, bend left=40, "f_*"]
\end{tikzcd}$$

corresponds to the following adjoint triple

$$\begin{tikzcd}
\mathrm{Fun}(L, \mathrm{Ani}) \arrow[r, "f^*"] & \mathrm{Fun}(K, \mathrm{Ani}) \arrow[l, bend right=40, "f_!"'] \arrow[l, bend left=40, "f_*"]
\end{tikzcd}$$

where $f_!$ resp. f_* are the left resp. right Kan extension of f along the Yoneda embedding. Therefore, every diagram $P\colon I \to \mathrm{Ani}$ naturally yields a diagram in $I^{\mathrm{op}} \to \mathrm{Cat}_\infty$ sending every transition map $f_{i\to j}\colon K_i \to K_j$ for $i \to j$ in I^{op} to f^*.

Thus, ϱ can pointwise for a $P \in \mathrm{Pro}(\mathrm{Ani})$ with presentation $I \xrightarrow{P} \mathrm{Ani}$ be calculated as

$$\varrho(P) = \varrho^*(P) \simeq \operatorname*{colim}_{I^{\mathrm{op}}} (\mathrm{Ani}_{/-})^* \circ P^{\mathrm{op}} \simeq \operatorname*{colim}_{i\in I^{\mathrm{op}}} \mathrm{Fun}(P_i, \mathrm{Ani}) = \mathrm{Fun}(P, \mathrm{Ani}).$$

where the small filtered colimit is taken $\widehat{\mathrm{Cat}}_\infty$ as the inclusion $\mathcal{LT}\mathrm{op} \subset \widehat{\mathrm{Cat}}_\infty$ preserves small filtered colimits.

Definition 3.3.11. Given an ∞-topos $\mathcal{X}$, we refer to $\varrho(\Pi_\infty \mathcal{X})$ as the ∞-topos of **local systems** on $\mathcal{X}$ and call an $l \in \varrho(\Pi_\infty \mathcal{X})$ a **local system** of $\mathcal{X}$.
Pick a cofiltered diagram $X\colon I \to \mathrm{Ani}$ such that $\Pi_\infty \mathcal{X} \simeq \lim_{i\in I} X_i$. A local system $l \in \varrho(\Pi_\infty \mathcal{X})$ is said to be **split**, if it lies in the essential image of the canonical map

$$\mathrm{Ani}_{/X_i} \xrightarrow{\iota^*_{X_i}} \varrho^*(\Pi_\infty \mathcal{X}) = \varphi(\Pi_\infty \mathcal{X}).$$

In this case, we say that X_i represents l.

Notation 3.3.12. Let $\tau_\bullet\mathrm{Ani} \subset \mathrm{Ani}$ be some full subcategory of Ani. By slight overload of notation, elements in $\mathrm{Fun}(\Pi_\infty \mathcal{X}, \tau_\bullet\mathrm{Ani})$ or $\mathrm{Fun}(\widehat{\Pi}^{(\mathrm{ab})}_\infty \mathcal{X}, \tau_\bullet\mathrm{Ani})$ are also called local systems. They are commonly identified with their essential image under the fully faithful functors

$$\mathrm{Fun}(\Pi_\infty \mathcal{X}, \tau_\bullet\mathrm{Ani}) \to \mathrm{Fun}(\Pi_\infty \mathcal{X}, \mathrm{Ani}) \quad \text{and} \quad \mathrm{Fun}(\widehat{\Pi}_\infty \mathcal{X}, \tau_\bullet\mathrm{Ani}) \to \mathrm{Fun}(\widehat{\Pi}_\infty \mathcal{X}, \mathrm{Ani}).$$

Thus, a local system $l \in \mathrm{Fun}(\Pi_\infty \mathcal{X}, \tau_\bullet\mathrm{Ani})$ or $l \in \mathrm{Fun}(\widehat{\Pi}^{(\mathrm{ab})}_\infty \mathcal{X}, \tau_\bullet\mathrm{Ani})$ is split if it lies in the essential image of

$$\mathrm{Fun}(X_i, \tau_\bullet\mathrm{Ani}) \to \mathrm{Fun}(\Pi_\infty \mathcal{X}, \tau_\bullet\mathrm{Ani}) \quad \text{resp.} \quad \mathrm{Fun}(X_i, \tau_\bullet\mathrm{Ani}) \to \mathrm{Fun}(\widehat{\Pi}^{(\mathrm{ab})}_\infty \mathcal{X}, \tau_\bullet\mathrm{Ani})$$

for some $i \in I$ for a presentation $I \xrightarrow{X} \mathrm{Ani}$ of $\Pi_\infty \mathcal{X}$ or $\widehat{\Pi}^{(\mathrm{ab})}_\infty \mathcal{X}$.

Each split local system can be written as a pullback of constant objects.

Observation 3.3.13. Let $l \in \varrho(\Pi_\infty \mathcal{X})$ be a split local system. By assumption, there is an anima K, which appears in a filtered diagram presenting $\Pi_\infty \mathcal{X}$, such that $l \simeq \iota^*_K(f)$ for some $f\colon M \to K$ in $\mathrm{Ani}_{/K}$ under the canonical $\mathrm{Ani}_K \xrightarrow{\iota^*} \varrho^*(\Pi_\infty \mathcal{X})$. Consider the canonical pullback square in $\mathrm{Ani}_{/K}$ (Obs. A.1.9)

$$\begin{tikzcd}
f \arrow[r] \arrow[d] & M \times K \arrow[d] \\
1_K \arrow[r] & K \times K
\end{tikzcd}$$

where $- \times K\colon \mathrm{Ani} \to \mathrm{Ani}_{/K}$ is the geometric morphism induced by $K \to *$. Since φ^* and ι^* preserves finite limits, this exhibits $\varphi^*(l)$ as the pullback

$$\begin{tikzcd}
\varphi^*(l) \simeq \varphi^*\iota^*(f) \arrow[r] \arrow[d] & \underline{M} \arrow[d] \\
1_{\mathcal{X}} \arrow[r, "f_{\varphi^*\iota^*}"] & \underline{K}
\end{tikzcd}$$

Here, we use that $\varphi^* \circ \iota^* \circ (- \times K) \simeq \underline{\Delta}$, since Ani is initial in $\mathcal{LT}\mathrm{op}$.

The following Proposition of Hoyois relates local systems to locally constant sheaves via the functors ϱ and ψ.

Proposition 3.3.14 ([Hoy17, Prop.3.4])**.** *Let $\mathcal{X}$ be a ∞-topos. Then,*

1) *For any split local system $l \in \varrho(\Pi_\infty \mathcal{X})$, the object $\varphi^*(l) \in \mathcal{X}$ is locally constant.*

2) *For a split local system $l \in \mathrm{Fun}(\Pi_\infty \mathcal{X}, \mathrm{Ani}^\pi)$, $\varphi^*(l)$ is **finite** locally constant.*

3) *Let $n \geq -2$. If $l \in \mathrm{Fun}(\Pi_\infty \mathcal{X}, \tau_{\leq n}\mathrm{Ani})$ is split, then $\varphi^*(l) \in \tau_{\leq n}\mathcal{X}^{\mathrm{lc}}$.*

4) *Let $l \in \mathrm{Fun}(\widehat{\Pi}_\infty \mathcal{X}, \mathrm{Ani}^\pi)$ be a split local system. Then, $\psi^*(l) = \varphi^* \circ i_\pi(l)$ is lissé.*

Prior to the proof of Prop. 3.3.14, recall that every local system is a filtered colimit of split local systems. Since φ^* and ψ^* preserve all colimits, the following is a direct consequence:

Corollary 3.3.15. *The essential image of $\varrho(\Pi_\infty \mathcal{X})$ under φ^* is given by filtered colimits of locally constant sheaves. Similarly, $\varphi^*(\mathrm{Fun}(\Pi_\infty \mathcal{X}, \mathrm{Ani}^\pi))$ and $\psi^*(\mathrm{Fun}(\widehat{\Pi}_\infty \mathcal{X}, \mathrm{Ani}^\pi))$ are given by filtered colimits of diagrams in in $\mathcal{X}^{\mathrm{flc}}$ resp. $\mathcal{X}^{\mathrm{liss\acute{e}}}$.*

The proof of Prop. 3.3.14 requires two preliminary results. The first, Lem. 3.3.16, is particularly relevant for the proof of the last statement of Prop. 3.3.14.

Lemma 3.3.16 ([BH20, p.54])**.** *Let $K \in \mathrm{Ani}$ and let $A, B \colon K \to \mathrm{Ani}$ be functors. For a $Y \in \mathrm{Pro(Ani)}$ with presentation $Y \simeq \lim_{i \in I} j(Y_i)$ and a functor $f \colon Y \to K$ i.e $f := \mathrm{colim}_{i \in I^{\mathrm{op}}} Y_i \xrightarrow{f_i} K$ there is an equivalence*

$$\mathrm{map}_{\mathrm{Fun}(Y,\mathrm{Ani})}(A \circ f, B \circ f) \simeq \mathrm{map}_{\mathrm{Pro(Ani)}\big/_{j(K)}}(Y, j(\underline{\mathrm{Hom}}_K(\textstyle\int A, \int B)))$$

where $\underline{\mathrm{Hom}}_K(\int A, \int B) \in {}^{\mathrm{Ani}}\big/_{K}$ is the internal mapping object (see Def. A.2.2) and $\int A \to K$, $\int B \to K$ are the anima obtained by applying the unstraightening functor $\int$ to A and B.

Proof. By Rmk. 3.3.10, there are the following equivalences for every $i \in I$:

$$\begin{aligned} \mathrm{map}_{\mathrm{Fun}(Y_i,\mathrm{Ani})}(f_i^* A, {f_i}^* B) &\overset{\text{unstraighten}}{\simeq} \mathrm{map}_{\mathrm{Ani}_{/Y_i}}(\textstyle\int f_i^* A, \int {f_i}^* B) \\ \overset{f_i^* \dashv (f_i)_*}{\simeq} \mathrm{map}_{\mathrm{Ani}_{/K}}(\textstyle\int A, (f_i)_* \circ {f_i}^* \int B) &\overset{\text{Lem. A.2.3}}{\simeq} \mathrm{map}_{\mathrm{Ani}_{/K}}(\textstyle\int A, \underline{\mathrm{Hom}}_K(Y_i, \int B) \\ \simeq \mathrm{map}_{\mathrm{Ani}_{/K}}(\textstyle\int A \times_K Y_i, \int B) &\simeq \mathrm{map}_{\mathrm{Ani}_{/K}}(Y_i, \underline{\mathrm{Hom}}_K(\textstyle\int A, \int B)) \end{aligned}$$

Postcomposing with A and B yields the functors

$$A_!, B_! \colon \mathrm{Cat}_{\infty/K} \to \mathrm{Cat}_{\infty/\mathrm{Ani}}.$$

As Cat_∞ has all pullbacks, $A_!$ and $B_!$ are left adjoints (see Cor. A.1.5). Thus, composing with

A and B commutes with small colimits. This yields the equivalences:

$$\begin{aligned}
&\operatorname{map}_{\operatorname{Fun}(Y,\mathrm{Ani})}(A\circ f, B\circ f)\\
&\simeq \operatorname{map}_{\operatorname{colim}_{i\in I^{\mathrm{op}}}\operatorname{Fun}(Y_i,\mathrm{Ani})}(\operatorname*{colim}_{i\in I^{\mathrm{op}}} f_i^* A, \operatorname*{colim}_{i\in I^{\mathrm{op}}} f_i^* B)\\
&\simeq \operatorname*{colim}_{i\in I^{\mathrm{op}}} \operatorname{map}_{\operatorname{Fun}(Y_i,\mathrm{Ani})}(f_i^* A, f_i^* B)\\
&\simeq \operatorname*{colim}_{i\in I^{\mathrm{op}}} \operatorname{map}_{\mathrm{Ani}_{/K}}(Y_i, \underline{\operatorname{Hom}}_K(\textstyle\int A, \int B)) = \operatorname*{colim}_{i\in I^{\mathrm{op}}} j^{\mathrm{op}}(Y_i)\,(\underline{\operatorname{Hom}}_K(\textstyle\int A, \int B))\\
&\simeq \lim_{i\in I} j(Y_i)\,(\underline{\operatorname{Hom}}_K \textstyle\int A, \int B) \simeq Y(\underline{\operatorname{Hom}}_K(\int A, \int B))\\
&\simeq \operatorname{map}_{\operatorname{Pro}(\mathrm{Ani})_{/jK}}(Y, j(\underline{\operatorname{Hom}}_K(\textstyle\int A, \int B)))
\end{aligned}$$

where we constantly identify $\operatorname{Pro}(\mathrm{Ani}_{/K})$ with $\operatorname{Pro}(\mathrm{Ani})_{/jK}$ in the last three lines. Note that by 3.1.3, $j^{\mathrm{op}}\colon \mathrm{Ani}_{/K}^{\mathrm{op}} \to \operatorname{Fun}(\mathrm{Ani}_{/K}, \mathrm{Ani})$ denotes the opposite Yoneda embedding and thus, $j(L)(M) \simeq j^{\mathrm{op}}(L)(M)$ for all $L, M \in \mathrm{Ani}_{/K}$. ☐

The previous Lemma provides us with a detailed description of the pro-functors $\operatorname{Fun}(\widehat{X}, \mathrm{Ani}^\pi)$ as pro-functors with domain X, which admit a finiteness condition.

Lemma 3.3.17 ([BH20, p.54])**.** *For every pro-anima* $X \in \operatorname{Pro}(\mathrm{Ani})$*, the unit* $X \xrightarrow{\eta_X} i_\pi \widehat{X}$ *induces an equivalence*

$$\operatorname{Fun}(\widehat{X}, \mathrm{Ani}^\pi) \xrightarrow{\simeq} \operatorname{Fun}_{\mathrm{fin}}(X, \mathrm{Ani}^\pi)$$

where $\operatorname{Fun}_{\mathrm{fin}}(X, \mathrm{Ani}^\pi)$ *is the full subcategory of* $\operatorname{Fun}(X, \mathrm{Ani}^\pi)$ *spanned by the functors with only finitely many values* Ani^π*.*

Proof. **Fully faithfulness:** First, consider the following sequence of equivalences, which stem directly from the definitions and are natural in $K \in \mathrm{Ani}^\pi$

$$\begin{aligned}
&\operatorname{map}_{\operatorname{Pro}(\mathrm{Ani})}(i_\pi \widehat{X}, j(K)) \overset{i_\pi \text{ fully faithful}}{\simeq} \operatorname{map}_{\operatorname{Pro}(\mathrm{Ani}^\pi)}(\widehat{X}, j(K))\\
&\overset{\text{Yoneda}}{\simeq} \widehat{X}(K) \overset{K\in \mathrm{Ani}^\pi}{\simeq} X(K) \overset{\text{Yoneda}}{\simeq} \operatorname{map}_{\operatorname{Pro}(\mathrm{Ani})}(X, j(K)).
\end{aligned}$$

This equivalence of mapping spaces carries over to an equivalence for all mapping spaces of slice categories. Indeed, for every $K \in \mathrm{Ani}^\pi$, $L \in \mathrm{Ani}$ and $\widehat{X} \xrightarrow{f} j(L)$, $j(K) \xrightarrow{g} j(L)$ in $\operatorname{Pro}(\mathrm{Ani}^\pi)_{/j(L)}$, we have

$$\begin{aligned}
&\operatorname{map}_{\operatorname{Pro}(\mathrm{Ani}^\pi)_{/j(L)}}(i_\pi \widehat{X}, j(K)) \overset{\text{Lem. A.1.2}}{\simeq} \operatorname{map}_{\operatorname{Pro}(\mathrm{Ani}^\pi)}(\widehat{X}, j(K)) \times_{\operatorname{map}_{\mathrm{Ani}}(K,L)} \{f\}\\
&\simeq \operatorname{map}_{\operatorname{Pro}(\mathrm{Ani})}(X, j(K)) \times_{\operatorname{map}_{\mathrm{Ani}}(K,L)} \{f\} \overset{\text{Lem. A.1.2}}{\simeq} \operatorname{map}_{\operatorname{Pro}(\mathrm{Ani})_{/j(L)}}(X, j(K))
\end{aligned}$$

Now, fix a presentation $I \xrightarrow{X} \mathrm{Ani}^\pi$ of $\widehat{X}$, i.e $\widehat{X} \simeq \lim_{i\in I} j(X_i)$. Pick out any $i \in I$ and let $\widehat{X} \xrightarrow{f_i} j(X_i)$ be the morphism associated to i. Let A, B be two parallel functors $A, B\colon X_i \to \mathrm{Ani}^\pi$. The unit $X \xrightarrow{\eta_X} i_\pi \widehat{X}$ then induces the equivalences

$$\begin{aligned}
&\operatorname{map}_{\operatorname{Fun}(\widehat{X},\mathrm{Ani}^\pi)}(A\circ f_i, B\circ f_i) \overset{\text{Lem. 3.3.16}}{\simeq} \operatorname{map}_{\operatorname{Pro}(\mathrm{Ani}^\pi)_{/j(X_i)}}(\widehat{X}, j(\underline{\operatorname{Hom}}_{X_i}(\textstyle\int A, \int B))\\
&\simeq \operatorname{map}_{\operatorname{Pro}(\mathrm{Ani})_{/j(X_i)}}(X, j(\underline{\operatorname{Hom}}_{X_i}(\textstyle\int A, \int B)) \overset{\text{Lem. 3.3.16}}{\simeq} \operatorname{map}_{\operatorname{Fun}(X,\mathrm{Ani})}(A\circ f_i\circ \eta_X, B\circ f_i \circ \eta_X)
\end{aligned}$$

where $\int$ denotes the unstraightening functor ([Lan21, 3.3.17]). Here the equivalence in the middle follows from the equivalence above, as $\underline{\mathrm{Hom}}_{X_i}(\int A, \int B) \in \mathrm{Ani}^{\pi}$. This is because $\int A, \int B \in \mathrm{Ani}^{\pi}_{/X_i}$, since, by construction of $\int$, all fibres of $\int A \to X_i$ and $\int B \to X_i$ are in Ani^{π} and $\pi_0(X_i)$ is finite.

Essential surjectivity: Pick a $F \in \mathrm{Fun}_{\mathrm{fin}}(X, \mathrm{Ani}^{\pi)}$. By the description of Pro(Ani) as right Kan extensions, there are functors $F_K := F_{(K, X \to j(K))} \colon K \to \mathrm{Ani}^{\pi}$ for every $(K, X \to j(K)) \in \mathrm{Ani}_{X/}$ such that each F_K takes only finitely many values and $F \simeq \mathrm{colim}_{(K,X \to j(K)) \in \mathrm{Ani}_{X/}} F_K$. Let $\mathcal{C}_K \subset \mathrm{Ani}^{\pi}$ denote the full subcategory spanned by the values of F_K. Since K is an ∞-groupoid, the image $F_K(K)$ is a sub-∞-groupoid of the groupoid core $\mathcal{C}_K^{\simeq}$. Notice, that $F_K(K)$ is itself an object of Ani^{π} as $\mathcal{C}_K^{\simeq}$ only has finitely many vertices. Composing the map $X \to j(K)$ of pro-anima with $j(F_K)$ yields a canonical map $X \to j(F_K(K))$. Thus, every $F_K(K)$ is an object of $\mathrm{Ani}^{\pi}_{X/}$, and hence participates in the colimit forming F. This means in particular, that it suffices to consider only the F_L for $L \in \mathrm{Ani}^{\pi}_{X/}$ to calculate F. However, profinite completion $\widehat{(-)}$ induces the equivalence $\mathrm{Ani}^{\pi}_{X/} \xrightarrow{\simeq} \mathrm{Ani}^{\pi}_{\widehat{X}/}$, since the composition

$$\mathrm{Ani}^{\pi} \subset \mathrm{Ani} \xrightarrow{j} \mathrm{Pro}(\mathrm{Ani}) \xrightarrow{\widehat{(-)}} \mathrm{Pro}(\mathrm{Ani}^{\pi})$$

is equivalent to $j \colon \mathrm{Ani}^{\pi} \to \mathrm{Pro}(\mathrm{Ani}^{\pi})$. In summary, the preceding analysis yields:

$$F \simeq \underset{(K,X \to j(K)) \in \mathrm{Ani}_{X/}}{\mathrm{colim}} F_K \simeq \underset{(K,X \to j(K) \in \mathrm{Ani}^{\pi}_{X/}}{\mathrm{colim}} F_K \simeq \underset{(K,\widehat{X} \to j(K)) \in \mathrm{Ani}^{\pi}_{\widehat{X}/}}{\mathrm{colim}} F_K.$$

By description of $\mathrm{Pro}(\mathrm{Ani}^{\pi})$ as right Kan extensions the right side is an element of $\mathrm{Fun}(\widehat{X}, \mathrm{Ani}^{\pi})$. □

Finally, we have established all tools to proof of Prop. 3.3.14.

Proof of Prop. 3.3.14. The proof for *2)* and *3)* differs from the proof of *1)* only by restricting to Ani^{π} at some places. We indicate the restrictions by a π in brackets.

Suppose the split local system $l \in \mathrm{Fun}(\Pi_{\infty} \mathcal{X}, \mathrm{Ani}^{(\pi)})$ is given by $M \in \mathrm{Ani}^{(\pi)}{}_{/K}$ under the canonical geometric morphism $\mathrm{Ani}_{/K} \xrightarrow{\iota^*} \varrho(\Pi_{\infty} \mathcal{X})$, as in Obs. 3.3.13. In particular there is a pullback diagram

$$\begin{tikzcd}
\varphi^*(l) \arrow[r] \arrow[d] & \underline{M} \arrow[d] \\
1_{\mathcal{X}} \arrow[r] & \underline{K}
\end{tikzcd}$$

Let $\coprod_{\alpha \in A} U_{\alpha} \to \mathrm{id}_K$ be an effective epimorphism in $\mathrm{Ani}_{/K}$ (i.e. a cover of K), where each U_{α} is contractible. Consider the following diagram in $\mathcal{LT}$op for an arbitrary $\alpha \in A$

$$\begin{tikzcd}
{}^{\mathrm{Ani}^{(\pi)}}\!\diagup_{K} \arrow[r, "\underline{\Delta}"] \arrow[d, "-\times_K U_\alpha"'] & {}^{\mathcal{X}}\!\diagup_{\underline{K}} \arrow[r, "-\times_{\underline{K}} 1_{\mathcal{X}}"] \arrow[d, "-\times_{\underline{K}} \underline{U_\alpha}"'] & {}^{\mathcal{X}}\!\diagup_{1_{\mathcal{X}}} \arrow[d, "-\times \varphi^*\iota^*(U_\alpha)"] \\
\mathrm{Ani}^{(\pi)} \simeq {}^{\mathrm{Ani}^{(\pi)}}\!\diagup_{U_\alpha} \arrow[r, "\underline{\Delta}"] & {}^{\mathcal{X}}\!\diagup_{\underline{U_\alpha}} \arrow[r, "-\times_{\underline{U_\alpha}} \varphi^*\iota^*(U_\alpha)"] & {}^{\mathcal{X}}\!\diagup_{\varphi^*\iota^*(U_\alpha)} \simeq \mathcal{X}
\end{tikzcd}$$

The right hand square commutes, because it is induced by the pullback square

$$\begin{tikzcd}
\varphi^*\iota^*(U_\alpha) \arrow[r] \arrow[d] & \underline{U_\alpha} \arrow[d] \\
1_{\mathcal{X}} \arrow[r] & \underline{K}
\end{tikzcd}$$

Likewise, the left hand square commutes, because the geometric morphism $\underline{\Delta}$ commutes with pullbacks. Moreover, $\mathrm{Ani} \simeq \mathrm{Ani}_{/U_\alpha}$ and $\mathcal{X} \simeq \mathcal{X}/\varphi^*\iota^*(U_\alpha)$ since U_α is contractible.

Following its path through the upper right corner in this diagram, an $M \in {}^{\mathrm{Ani}^{(\pi)}}\!\diagup_{K}$ from the upper left corner is sent to

$$\underline{M} \times_{\underline{K}} 1_{\mathcal{X}} \times \varphi^*\iota^*(U_\alpha) \in \mathcal{X}$$

in the lower right corner. However, we had chosen M and K such that $\underline{M} \times_{\underline{K}} 1_{\mathcal{X}} \simeq \varphi^*(l)$. Thus, commutativity of the diagram yields

$$\underline{\Delta}(M \times_K U_\alpha) \times \varphi^*\iota^*(U_\alpha) \simeq \varphi^*(l) \times \varphi^*\iota^*(U_\alpha)$$

where the left hand side is the obtained by the path through the lower left corner. This makes $\varphi^*(l) \times \varphi^*\iota^*(U_\alpha)$ a constant object of $\mathcal{X}_{/\varphi^*\iota^*(U_\alpha)}$, which proves *1)*, since $\coprod_\alpha \varphi^*\iota^*(U_\alpha) \to \varphi^*\iota^*(\mathrm{id}_K) \simeq 1_{\mathcal{X}}$ is an effective epimorphism.
In the setting of *2)* and *3)*, one notices that $M \times_K U_\alpha \in \mathrm{Ani}^\pi$ resp. $\in \tau_{\leq n}\mathrm{Ani}$ in the above equivalence, since in this case M was assumed to be π-finite resp. n-truncated and U_α is contractible.

At *4)*, we are in the situation that $l \in \mathrm{Fun}(\widehat{\Pi}_\infty \mathcal{X}, \mathrm{Ani}^\pi) \overset{\text{Lem. 3.3.17}}{\simeq} \mathrm{Fun}_{\mathrm{fin}}(\widehat{\Pi}_\infty \mathcal{X}, \mathrm{Ani}^\pi)$. Therefore, the element $M \to K$ of $\mathrm{Ani}^\pi_{/K}$ that represents l only has finitely many non-equivalent fibers, since applying the straigthening functor Str to $M \to K$ yields a functor

$$\mathrm{Str}(M)\colon K \to \mathrm{Ani}^\pi, \quad k \mapsto M_k$$

which only takes finitely many values (cf.[Lan21, 3.3.17]). Now, the previously chosen effective epimorphism $\coprod_{\alpha \in A} U_\alpha \to K$ pulls back to the effective epimorphism $\coprod_{\alpha \in A} U_\alpha \times_K M \to M$. The image of each U_α in K identifies with a point $k_\alpha \in K$, since U_α is contractible. In conclusion, there is a finite subset $A^{\mathrm{fin}} \subset A$ such that for every $\alpha \in A$ the fibre M_{k_α} is equivalent to an M_{k_β} for $\beta \in A^{\mathrm{fin}}$. Therefore,

$$\coprod_{\alpha \in A^{\mathrm{fin}}} U_\alpha \times_K M \simeq \coprod_{\alpha \in A^{\mathrm{fin}}} M_{k_\alpha} \simeq \coprod_{\alpha \in A} M_{k_\alpha} \simeq \coprod_{\alpha \in A} U_\alpha \times_K M \to M$$

is an effective epimorphism. This carries over to an effective epimorphism

$$\coprod_{\alpha\in A^{\mathrm{fin}}} \varphi^*\iota^*(M)\times\varphi^*\iota^*(U_\alpha)\to\varphi^*(l)$$

which exhibits $\varphi^*(l)$ as element of $\mathcal{X}^{\mathrm{lissé}}$, as $\underline{\Delta}(M\times_K U_\alpha)\times\varphi^*\iota^*(U_\alpha)\simeq\varphi^*(l)\times\varphi^*\iota^*(U_\alpha)$ by the above. □

However, φ^* and ψ^* not only send split local systems to locally constant objects, but all locally constant objects arise in this way.

Proposition 3.3.18. *Let $\mathcal{X}$ be an ∞-topos. The full subcategory $\mathcal{X}^{\mathrm{lc}}\subset\mathcal{X}$ is contained in the essential image of $\varphi^*\colon\varrho(\Pi_\infty\mathcal{X})\to\mathcal{X}$. This result admits the following refinements:*

1) *$\tau_{\leq n}\mathcal{X}^{\mathrm{lc}}$ is contained in the essential image of $\mathrm{Fun}(\Pi_\infty\mathcal{X},\tau_{\leq n}\mathrm{Ani})\xrightarrow{\varphi^*}\tau_{\leq n}\mathcal{X}$ for any $n\geq -2$.*
2) *$\mathcal{X}^{\mathrm{flc}}$ is contained in the essential image of $\mathrm{Fun}(\Pi_\infty\mathcal{X},\mathrm{Ani}^\pi)\xrightarrow{\varphi^*}\mathcal{X}$.*
3) *$\mathcal{X}^{\mathrm{lissé}}$ is contained in the essential image of $\mathrm{Fun}(\widehat{\Pi}_\infty\mathcal{X},\mathrm{Ani}^\pi)\xrightarrow{\varphi^*}\mathcal{X}$.*

Proof. Pick any $X\in\mathcal{X}^{\mathrm{lc}}$ and let $\coprod_{\alpha\in A}U_\alpha\xrightarrow{\sqcup_\alpha p_\alpha}1_{\mathcal{X}}$ be an effective epimorphism, such that $X\times U_\alpha\simeq\underline{K_\alpha}\times U_\alpha$ for some $K_\alpha\in\mathrm{Ani}$. Note that $X\simeq\operatorname*{colim}_{\mathrm{N}(\Delta^{\mathrm{op}})}\check{\mathrm{C}}(\coprod_{\alpha\in A}\underline{K_\alpha}\times U_\alpha\to X)$. Since $\mathrm{Fun}(\Pi_\infty\mathcal{X},\mathrm{Ani})$ contains all small colimits, it suffices to show, that each $\underline{K_\alpha}\times U_\alpha$ is in the essential image of $\mathrm{Fun}(\Pi_\infty\mathcal{X},\mathrm{Ani})$.
Thus, pick an $\alpha\in A$ and observe that $p_\alpha\colon U_\alpha\to 1_{\mathcal{X}}$ induces a map

$$p_\alpha^*\colon\ \Pi_\infty\mathcal{X}=\Pi_\infty^{\mathcal{X}}1_{\mathcal{X}}\to\Pi_\infty^{\mathcal{X}}U_\alpha\simeq\Pi_\infty\mathcal{X}_{/U_\alpha}$$

in $\mathrm{Pro}(\mathrm{Ani})^{\mathrm{op}}\subset\mathrm{Fun}(\mathrm{Ani},\mathrm{Ani})$. Applying the functor ϱ^* of Rmk. 3.3.8 yields a commutative square in $\mathcal{LT}\mathrm{op}$

$$\begin{tikzcd}
\varrho(\Pi_\infty\mathcal{X}) \arrow[r, "\varphi^*_{\mathcal{X}}"] \arrow[d, "\varrho^*(p_\alpha^*)"'] & \mathcal{X} \arrow[d, "-\times U_\alpha"] \\
\varrho(\Pi_\infty\mathcal{X}_{/U_\alpha}) \arrow[r, "\varphi^*_{U_\alpha}"] & \mathcal{X}_{/U_\alpha}
\end{tikzcd}$$

By Definition, $X\times U_\alpha\simeq\underline{K_\alpha}\times U_\alpha\simeq\underline{\Delta}_{\mathcal{X}_{/U_\alpha}}$ in $\mathcal{X}_{/U_\alpha}$ and therefore $X\times U_\alpha\simeq\varphi^*_{U_\alpha}\underline{K_\alpha}$. Now, $X\times U_\alpha\in\mathcal{X}$ is the image of $X\times U_\alpha\in\mathcal{X}_{/U_\alpha}$ under the forgetful map $t_!$. This is the left adjoint to the étale map $-\times U_\alpha$, thus participates in adjoint triple. Since ϱ naturally factors through $\mathcal{RT}\mathrm{op}^{\mathrm{ess}}$ (Obs. 3.3.5), $\varrho^*(p_\alpha^*)$ has left adjoint $\varrho_!(p_\alpha^*)$ and the following square commutes

$$\begin{tikzcd}
\varrho(\Pi_\infty\mathcal{X}) \arrow[r, "\varphi^*_{\mathcal{X}}"] & \mathcal{X} \\
\varrho(\Pi_\infty\mathcal{X}_{/U_\alpha}) \arrow[u, "\varrho_!(p_\alpha^*)"'] \arrow[r, "\varphi^*_{U_\alpha}"] & \mathcal{X}_{/U_\alpha} \arrow[u, "t_!"]
\end{tikzcd}$$

Therefore,

$$X \times U_\alpha \simeq t_! \circ \varphi^*_{U_\alpha}(\underline{K_\alpha}) \simeq \varphi^*_{\mathcal{X}} \circ \varrho_!(p^*_\alpha) \circ \underline{\Delta}_{\varrho(\Pi_\infty \mathcal{X}_{/U_\alpha})}(K_\alpha)) \simeq \varphi^*_{\mathcal{X}} \circ \varrho_!(p^*_\alpha) \circ \varrho^*(p^*_\alpha)\,\underline{\Delta}_{\mathcal{X}}(K_\alpha).$$

By the proof of Prop. 3.3.14 the maps $\tau_{\leq n}\mathrm{Ani} \xrightarrow{\underline{\Delta}} \varrho(\Pi_\infty \mathcal{X}) \simeq \mathrm{Fun}(\Pi_\infty \mathcal{X}, \mathrm{Ani})$ and $\mathrm{Ani}^\pi \xrightarrow{\underline{\Delta}} \varrho(\Pi_\infty \mathcal{X})$ factor through $\mathrm{Fun}(\Pi_\infty \mathcal{X}, \tau_{\leq n}\mathrm{Ani})$ and $\mathrm{Fun}(\Pi_\infty \mathcal{X}, \mathrm{Ani}^\pi)$, respectively. Thus, the restrictions *1)* and *2)* readily follow.
The image of the fully faithful functor

$$\mathrm{Fun}(\widehat{\Pi}_\infty \mathcal{X}, \mathrm{Ani}^\pi) \overset{3.3.17}{\simeq} \mathrm{Fun}_{\mathrm{fin}}(\Pi_\infty \mathcal{X}, \mathrm{Ani}^\pi) \to \mathrm{Fun}(\Pi_\infty \mathcal{X}, \mathrm{Ani}) \simeq \varrho^*(\mathcal{X}) = \varrho(\mathcal{X})$$

consists of those $Y \in \varrho(\mathcal{X})$, which are finite filtered colimits of split local systems $l \in \mathrm{Fun}(\Pi_\infty \mathcal{X}, \mathrm{Ani}^\pi)$. By *2)* each $\underline{K}_\alpha \times U_\alpha$ is the image of a split $l_\alpha \in \mathrm{Fun}(\Pi_\infty \mathcal{X}, \mathrm{Ani}^\pi)$, if X is in $\mathcal{X}^{\mathrm{liss\acute{e}}}$. In this case, the indexing set A is assumed to be finite, so

$$Y := \mathrm{colim}_{[n] \in \mathrm{N}(\Delta^{\mathrm{op}})} \coprod_{(\alpha_1, \ldots, \alpha_n) \in A^n} l_{\alpha_1} \times \cdots \times l_{\alpha_n}$$

is a finite colimit, hence $Y \in \mathrm{Fun}(\widehat{\Pi}_\infty \mathcal{X}, \mathrm{Ani}^\pi)$. Then, $X \simeq \psi^*(Y)$ by construction. □

The main Theorems of this section propagate that the restrictions φ^* and ψ^* are equivalences in the correct settings.

Theorem 3.3.19 ([HA, Thm.A.1.15], [Hoy17, Thm.3.13])**.** *Let $-2 \leq n \leq \infty$ and $\mathcal{X}$ be an locally n-connected ∞-topos. Then, the geometric morphism $\varphi^*\colon \varrho(\Pi_\infty \mathcal{X}) \to \mathcal{X}$ induces a fully faithful functor*

$$\mathrm{Fun}(\Pi_\infty \mathcal{X}, \tau_{\leq n}\mathrm{Ani}) \to \tau_{\leq n}\varrho\,\Pi_\infty\mathcal{X} \to \tau_{\leq n}\mathcal{X}$$

whose essential image is the full subcategory of locally constant, n-truncated objects in $\mathcal{X}$ i.e. In case $n = \infty$, Obs. 3.3.5 implies that there is an equivalence

$$\mathrm{Ani}_{/\Pi_{\mathrm{const}}(1_{\mathcal{X}})} \simeq \mathcal{X}^{\mathrm{lc}}.$$

Quite remarkably, there no assumptions on $\mathcal{X}$ are needed, if one restricts to π-finite objects.

Theorem 3.3.20 ([Hoy17, Thm.4.3], [BH20, 10.1])**.** *For any ∞-topos $\mathcal{X}$, the geometric morphism $\varphi^*\colon \varrho(\Pi_\infty \mathcal{X}) \to \mathcal{X}$ induces equivalences*

$$1)\quad \mathrm{Fun}(\Pi_\infty(\mathcal{X}), \mathrm{Ani}^\pi) \simeq \mathcal{X}^{\mathrm{flc}} \qquad\qquad 2)\quad \mathrm{Fun}(\widehat{\Pi}_\infty \mathcal{X}, \mathrm{Ani}^\pi) \simeq \mathcal{X}^{\mathrm{liss\acute{e}}}.$$

To simplify the argument given in [Hoy17], we present a reduction which is used to establish both Thm. 3.3.19 and Thm. 3.3.20.

Lemma 3.3.21 ([Hoy17, p.11])**.** *Let $\mathcal{X}$ be an ∞-topos. Denote $\tau_\bullet\mathrm{Ani} \in \{\mathrm{Ani}, \tau_{\leq n}\mathrm{Ani}, \mathrm{Ani}^\pi\}$, where $0 \leq n < \infty$. Suppose that for every $U \in \mathcal{X}$ and for every $A, B \in \tau_\bullet\mathrm{Ani}$ the unit $\varphi^*_{\mathcal{X}_{/U}}$*

induces an equivalence

$$\mathrm{map}_{\mathrm{Fun}(\Pi_\infty \mathcal{X}_{/U}, \tau_\bullet \mathrm{Ani})}(X, Y) \xrightarrow{\simeq} \mathrm{map}_{\mathcal{X}_{/U}}(\underline{A}, \underline{B})$$

where $X, Y \colon \Pi_\infty \mathcal{X}_{/U} \to * \overset{A}{\underset{B}{\rightrightarrows}} \mathrm{Ani}^\pi$ *are the constant functors associated to the constant objects* $\underline{K}, \underline{L} \in \varrho(\mathcal{X}_{/U})$ *via* $\varrho(\mathcal{X}_{/U}) \simeq \mathrm{Fun}(\Pi_\infty \mathcal{X}_{/U})$*.*

Then, $\varphi^* \colon \mathrm{Fun}(\Pi_\infty \mathcal{X}, \tau_\bullet \mathrm{Ani}) \to \mathcal{X}$ *is fully faithful.*

Proof. Let $F \colon I \to \mathrm{Ani}$ be a cofiltered diagram, such that $\Pi_\infty \mathcal{X} \simeq \lim_{i \in I} F_i$. By Definition and Rmk. 3.3.10, $\mathrm{Fun}(\Pi_\infty \mathcal{X}, \tau_\bullet \mathrm{Ani})$ is precisely the colimit $\mathrm{colim}_{i \in I^{\mathrm{op}}} \tau_\bullet \mathrm{Ani}_{/F_i}$. Thus, it suffices to show that the induced map

$$\tau_\bullet \mathrm{Ani}_{/F_i} \to \mathrm{Ani}_{/F_i} \xrightarrow{\iota_i^*} \varrho(\Pi_\infty \mathcal{X}) \xrightarrow{\varphi^*} \mathcal{X}$$

is fully faithful for all $i \in I$.
Thus, pick a $i \in I$ and consider any two $K, L \in \tau_\bullet \mathrm{Ani}_{/F_i}$ and let $X, Y \in \mathrm{Fun}(F_i, \tau_\bullet \mathrm{Ani})$ denote the corresponding functors obtained by straightening K and L. Since $F_i \simeq \mathrm{colim}_{a \in F_i} *$, the terminal object id_{F_i} can be written as $\mathrm{colim}_{a \in F_i} * \xrightarrow{a} F_i$. Thus, there is a diagram

$$P \colon F_i \to \mathrm{Ani}_{/F_i} \xrightarrow{\iota_i^*} \varrho(\mathcal{X}) \xrightarrow{\varphi^*} \mathcal{X}$$

with colimit $1_{\mathcal{X}}$. Since Π_∞ is a left adjoint, this yields $\Pi_\infty \mathcal{X} = \Pi_\infty 1_{\mathcal{X}} \simeq \mathrm{colim}_{a \in F_i} \Pi_\infty P(a)$ in $\mathrm{Pro}(\mathrm{Ani})$. Now, fix an $a \in F_i$. The unique map $P(a) \to 1_{\mathcal{X}}$ induces a map $q_a \colon \Pi_\infty \mathcal{X} \to \Pi_\infty \mathcal{X}_{/P(a)}$ in $\mathrm{Pro}(\mathrm{Ani})^{\mathrm{op}} \subset \mathrm{Fun}(\mathrm{Ani}, \mathrm{Ani})$ and thus yields a functor p_a and a geometric morphism $\varrho^*(q_a)$ (see.Rmk 3.3.8) such that the following diagram commutes

$$\begin{tikzcd}
\mathrm{Fun}(\Pi_\infty \mathcal{X}, \tau_\bullet \mathrm{Ani}) \arrow[r, "p_a"] & \mathrm{Fun}(\Pi_\infty \mathcal{X}_{/P(a)}, \tau_\bullet \mathrm{Ani}) \\
\subset & \subset \\
\varrho(\Pi_\infty \mathcal{X}) \arrow[r, "\varrho^*(q_a)"] & \varrho(\Pi_\infty \mathcal{X}_{/P(a)})
\end{tikzcd}$$

Since ϱ is a right adjoint and ϱ^* is anti-equivalent to ϱ, the maps $\varrho^*(q_a)$ yield an equivalence

$$\varrho \, \Pi_\infty \mathcal{X}) \simeq \mathrm{colim} \, a \in F_i \varrho(\Pi_\infty \mathcal{X}_{/P(a)}$$

Thus, the p_a provide an equivalence

$$\mathrm{Fun}(\Pi_\infty \mathcal{X}, \tau_\bullet \mathrm{Ani}) \simeq \mathrm{Fun}(\Pi_\infty \mathcal{X}_{/P(a)}, \tau_\bullet \mathrm{Ani}).$$

Thus it suffices to prove that φ^* induces the following equivalence for every $a \in F_i$:

$$\mathrm{map}_{\mathrm{Fun}(\Pi_\infty \mathcal{X}_{/P(a)}, \tau_\bullet \mathrm{Ani})}(p_a \iota_i^*(X), p_a \iota_i^*(Y)) \simeq \mathrm{map}_{\mathcal{X}_{/P_a}}(\varphi^* p_a \iota_i^*(K), \varphi^* p_a \iota_i^*(L)).$$

Consider the following commutative square, induced by the canonical maps

$$\begin{tikzcd}
\mathrm{Fun}(F_i, \tau_\bullet \mathrm{Ani}) \arrow[r, "a^*"] \arrow[d, "\iota_i^*"] & \mathrm{Fun}(*, \tau_\bullet \mathrm{Ani}) \simeq \tau_\bullet \mathrm{Ani} \arrow[d, "\underline{\Delta}"] \\
\mathrm{Fun}(\Pi_\infty \mathcal{X}, \tau_\bullet \mathrm{Ani}) \arrow[r, "p_a"] & \mathrm{Fun}(\Pi_\infty P_a, \tau_\bullet \mathrm{Ani})
\end{tikzcd}$$

This square exhibits $p_a \iota_i^*(K)$ and $p_a \iota_i^*(L)$ as constant functors in $\mathrm{Fun}(\Pi_\infty \mathcal{X}_{/P_a}, \tau_\bullet \mathrm{Ani})$ and thus the assumption yields the desired equivalence. ☐

As most of the work has been outsourced, the proof of Thm. 3.3.19 mainly consists in putting together the pieces of the puzzle.

Proof of Thm. 3.3.19. Let $-2 \leq n \leq \infty$. By assumption $\mathcal{X}$ is locally n-connected. For an $U \in \mathcal{X}$ the composition

$$\tau_{\leq n}\mathrm{Ani} \xrightarrow{\tau_{\leq n} \underline{\Delta}_{\mathcal{X}}} \tau_{\leq n}\mathcal{X} \xrightarrow{\tau_{\leq n}(-\times U)} \tau_{\leq n}\mathcal{X}_{/U}$$

is equivalent to $\tau_{\leq n} \underline{\Delta}_{\mathcal{X}_{/U}}$. Since both $\tau_{\leq n} \underline{\Delta}_{\mathcal{X}}$ and $- \times U$ each have a left adjoint, $\mathcal{X}_{/U}$ is locally n-connected for any $U \in \mathcal{X}$. Therefore Cor. 2.3.29 provides an equivalence

$$\underline{\Delta}(\mathrm{map}_{\mathrm{Ani}}(X, Y)) \simeq \underline{\mathrm{Hom}}_{\mathcal{X}_{/U}}(\underline{X}, \underline{Y})$$

for any $X, Y \in \mathrm{Ani}_{\leq n}$, since $\underline{\Delta}\colon \mathrm{Ani}_{\leq n} \to (\mathcal{X}_{/U})_{\leq n}$ preserves all small limits. Taking global sections, i.e. applying $\mathrm{map}_{\mathcal{X}}(U, -)$, this yields for any $X, Y \in \mathrm{Ani}_{\leq n}$:

$$\begin{aligned}
&\mathrm{map}_{\mathcal{X}_{/U}}(\underline{X}, \underline{Y}) \simeq \mathrm{map}_{\mathcal{X}_{/U}}(1_{\mathcal{X}_{/U}}, \underline{\mathrm{Hom}}_{\mathcal{X}_{/U}}(\underline{X}, \underline{Y})) \simeq \Pi_\infty \mathcal{X}_{/U}(\mathrm{map}_{\mathrm{Ani}}(X, Y)) \\
&\simeq \mathrm{map}_{\mathrm{Pro(Ani)}}(\Pi_\infty \mathcal{X}_{/U}, j(\mathrm{map}_{\mathrm{Ani}}(X, Y))) \overset{\text{Lem. 3.3.16}}{\simeq} \mathrm{map}_{\mathrm{Fun}(\Pi_\infty U, \mathrm{Ani}_{\leq n})}(\mathrm{const} X, \mathrm{const}(Y))
\end{aligned}$$

Here $\mathrm{const}(X)$ and $\mathrm{const}(Y)$ are the constant functors

$$\Pi_\infty U \to * \underset{Y}{\overset{X}{\rightrightarrows}} \mathrm{Ani}_{\leq n}.$$

which are obtained from X, Y by straightening (see Rmk. 3.3.10). The statement of Thm. 3.3.19 now follows from Lem. 3.3.21. ☐

Lemma 3.3.22 ([Hoy17, 4.2])**.** *Let* $\tau_{<\infty}\colon \mathrm{Pro(Ani)} \to \mathrm{Pro}(\tau_{<\infty}\mathrm{Ani})$ *denote the canonical map restricting an* $X\colon \mathrm{Ani} \to \mathrm{Ani}$ *to* $X|_{\tau_{<\infty}\mathrm{Ani}}$. *For any ∞-topos $\mathcal{X}$ and any coherent* $K \in \mathrm{Ani}$*, there is a map* $\Pi_\infty \mathcal{X}_{/\underline{K}} \to K \times \Pi_\infty \mathcal{X}$*, which induces an equivalence*

$$\tau_{<\infty} \Pi_\infty \mathcal{X}_{/\underline{K}} \simeq \tau_{<\infty}(K \times \Pi_\infty \mathcal{X}).$$

Proof. First of all, consider the equivalence

$$\begin{aligned}
\Pi_\infty \mathcal{X}_{/\underline{K}} &= \mathrm{map}_{\mathcal{X}_{/\underline{K}}}(1_{\mathcal{X}_{/\underline{K}}}, \underline{\Delta}(-)) \overset{\text{A.1.3}}{\simeq} \mathrm{map}_{\mathcal{X}}(\underline{K}, \underline{\Delta}(-)) \\
&\simeq \mathrm{map}_{\mathcal{X}}(\underline{K} \times 1_{\mathcal{X}}, \underline{\Delta}(-)) \overset{2.3.28}{\simeq} \mathrm{map}_{\mathcal{X}}(\mathrm{colim}_K 1_{\mathcal{X}}, \underline{\Delta}(-)) \simeq \lim_{K^{\mathrm{op}}} \Pi_\infty \mathcal{X}.
\end{aligned}$$

where the limit is taken in Fun(Ani, Ani). Thus, the term above is equivalent to

$$\operatorname{colim}_K \Pi_\infty \mathcal{X} \in \mathrm{Pro}(\mathrm{Ani}) \subset \mathrm{Fun}(\mathrm{Ani}, \mathrm{Ani})^{\mathrm{op}}.$$

Now, fix a cofiltered diagram $X\colon I \to \mathrm{Ani}$ such that $\lim_{i\in I} j(X_i) \simeq \Pi_\infty \mathcal{X}$. By universal property of the colimit, there is a canonical map

$$\operatorname{colim}_K \Pi_\infty \mathcal{X} \simeq \operatorname{colim}_K \lim_I X_i \to \lim_I \operatorname{colim}_K X_i \overset{2.3.28}{\simeq} \lim_I K \times X_i \simeq K \times \Pi_\infty \mathcal{X}.$$

It is a standard fact, that for a coherent $A \in \mathrm{Ani}$, the functor $\mathrm{map}(A, -)|_{\tau_{\leq n}\mathrm{Ani}}\colon \tau_{\leq n}\mathrm{Ani} \to \mathrm{Ani}$ preserves filtered colimits for any $n \geq 0$ (cf. [DAGXIII, 2.3.9]). Since colim_K is the tensor product with anima i.e. left adjoint to the mapping space functor (cf.Lem. A.3.2), this provides the following equivalences for any $L \in \tau_{<\infty}\mathrm{Ani}$:

$$\begin{aligned}
&\mathrm{map}_{\mathrm{Pro(Ani)}}(\operatorname{colim}_K \Pi_\infty \mathcal{X}, j(L)) \overset{A.3.2}{\simeq} \mathrm{map}_{\mathrm{Ani}}(K, \mathrm{map}_{\mathrm{Pro(Ani)}}(\Pi_\infty \mathcal{X}, j(L)))\\
\overset{j(L)\text{ cocompact}}{\simeq}\ &\mathrm{map}_{\mathrm{Ani}}(K, \operatorname{colim}_I \mathrm{map}_{\mathrm{Ani}}(X_i, L) \overset{\text{[DAGXIII, 2.3.9]}}{\simeq} \operatorname{colim}_I \mathrm{map}_{\mathrm{Ani}}(K, \mathrm{map}_{\mathrm{Ani}}(X_i, L))\\
\overset{2.3.28}{\simeq}\ &\operatorname{colim}_I \mathrm{map}_{\mathrm{Ani}}(K \times X_i, L) \simeq \mathrm{map}_{\mathrm{Pro(Ani)}}(K \times \Pi_\infty \mathcal{X}, L).
\end{aligned}$$

where the third equivalence persists since $j(L)$ is cocompact in Pro(Ani) as $y^{\mathrm{op}}(L)$ is compact in $\mathrm{Ind}(\mathrm{Ani}^{\mathrm{op}})$. ☐

Proof of Thm. 3.3.20. 1. Let $A, B \in \mathrm{Ani}^\pi$. Then, the constant objects $\underline{A}, \underline{B} \in \varrho(\mathcal{X})$ correspond to the constant functors $X, Y\colon \Pi_\infty \mathcal{X} \to * \overset{A}{\underset{B}{\rightrightarrows}} \mathrm{Ani}^\pi$ (see Rmk.) The previous results yield the following series of equivalences

$$\begin{aligned}
&\mathrm{map}_{\mathrm{Fun}(\Pi_\infty \mathcal{X}, \mathrm{Ani})}(X, Y) \overset{\text{Lem. 3.3.16}}{\simeq} \mathrm{map}_{\mathrm{Pro(Ani)}}(\Pi_\infty \mathcal{X}, j(\mathrm{map}_{\mathrm{Ani}}(A, B)))\\
\overset{j\simeq y^{\mathrm{op}}}{\simeq}\ &\mathrm{map}_{\mathrm{Pro(Ani)}}\, \mathrm{map}_{\mathrm{Pro(Ani)}}(\Pi_\infty \mathcal{X}, \underline{\mathrm{Hom}}(j(A), j(B))) \simeq \mathrm{map}_{\mathrm{Pro(Ani)}}(A \times \Pi_\infty \mathcal{X}, j(B))\\
\overset{\text{Lem. 3.3.22}}{\simeq}\ &\mathrm{map}_{\mathrm{Pro(Ani)}}(\Pi_\infty \mathcal{X}_{/\underline{\Delta}(A)}, j(B)) \simeq \Pi_\infty \mathcal{X}_{/\underline{\Delta}(A)}(B)\\
=\ &\mathrm{map}_{\mathcal{X}_{/\underline{\Delta}(A)}}(1_{\mathcal{X}_{/\underline{\Delta}(A)}}, \underline{B}) \overset{A.1.3}{\simeq} \mathrm{map}_{\mathcal{X}}(\underline{A}, \underline{B}).
\end{aligned}$$

This equivalence holds for all ∞-topoi, as we did not make any assumptions for $\mathcal{X}$ in this proof. Therefore, equivalence *1)* is derived from Lem. 3.3.21.

2) The functor ψ^* is equivalent to the composition

$$\mathrm{Fun}(\widehat{\Pi}_\infty \mathcal{X}, \mathrm{Ani}^\pi) \overset{\text{Lem. 3.3.17}}{\simeq} \mathrm{Fun}_{\mathrm{fin}}(\Pi_\infty \mathcal{X}, \mathrm{Ani}^\pi) \subset \mathrm{Fun}(\Pi_\infty \mathcal{X}, \mathrm{Ani}^\pi) \xrightarrow{\varphi^*} \mathcal{X}.$$

Hence, ψ^* is fully faithful by *1).* Therefore, Prop. 3.3.18 yields the desired equivalence. ☐

3.4 Locally Connected ∞-Topoi

Starting with a brief outline of the classical notion locally connected categories, we introduce locally n-connected ∞-topoi as such, where $\Pi_\infty \mathcal{X}$ is an actual left-adjoint up to degree n.

Definition 3.4.1. Let $\mathcal{C}$ be an ∞-category with (small) coproducts and an initial object. An object $c \in \mathcal{C}$ is **connected**, if $c \simeq c_1 \coprod c_2$ for $c_1, c_2 \in \mathcal{C}$ implies that c_1 or c_2 is an initial object. $\mathcal{C}$ is **locally connected**, if every object in $\mathcal{C}$ is a coproduct of connected objects.

Note. This retrieves the classical notion of locally connectedness for 1-categories (cf.[AM69]).

Example 3.4.2. For a locally connected topological space T, the ∞-topos $\mathrm{Shv}(T)$) is locally connected. For every locally Noetherian scheme X the small étale site $X_{\acute{e}t}$ is locally connected by [AM69].

Observation 3.4.3. An ∞-topos $\mathcal{X}$ is locally connected if and only if $\mathrm{Disc}(\mathcal{X})$ is locally connected, since the truncation $\tau_{\leq 0}$ is a left adjoint, hence preserves coproducts and initial objects and

Classically, the connected objects in sufficiently nice categories are precisely those that corepresent coproduct preserving functors (cf.[AM69, p.111]). As Carchedi shows, this remains valid more generally.

Proposition 3.4.4 ([Car16, 3.26])**.** *Let $\mathcal{C}$ be a locally connected ∞ with all (small) coproducts and pullbacks. Assume $\mathcal{C}$ is distributive i.e. coproducts commute with pullbacks. Then, an object $c \in \mathcal{C}$ is connected if and only if* $\mathrm{map}_{\mathcal{C}}(c, -) : \mathcal{C} \to \mathrm{An}$ *preserves coproducts.*

In classical topos theory, a locally connected 1-topoi is characterized by the existence of an actual left-adjoint representing the its shape.

Proposition 3.4.5 ([Joh02, C.3.3.6])**.** *A 1-topos $\mathcal{X}$ is locally connected if and only if the functor $\underline{\Delta}_{\mathcal{X}} : \mathrm{Set} \to \mathcal{X}$ admits a left adjoint $\Pi_0^{\mathcal{X}}$ given by $\Pi_0^{\mathcal{X}}(X) := I_X$, where I_X is the index set of the decomposition $X = \coprod_{i \in I_X}$ into connected objects $X_i \in \mathcal{X}$.*

4 From classical to ∞-Cohomology

This section develops a generalized notion of cohomology for sheaves with values in anima, hereafter referred to as ∞-cohomology. While recovering the classical abelian and non-abelian theory in the discrete case, this framework provides a natural generalization to higher-categorical settings. After establishing the foundational properties of Eilenberg–MacLane objects and ∞-bundles, we employ the theory of ∞-gerbes of [NSS14, §4] and [HTT, §7.2.2] to prove Theorem 4.6.2, which serves as central technical tool for the remainder of this work. This theorem establishes a robust, systematic lifting procedure: it permits the comparison of ∞-cohomology by leveraging data from classical abelian cohomology and principal bundles. Notably, since the shape $\Pi_\infty \mathcal{X}$ of an ∞-topos $\mathcal{X}$ characterized by cohomology with constant coefficients, we recover comparisons of shapes as a natural specialization of this general lifting result.

4.1 Eilenberg-MacLane objects and ∞-Cohomology

For an anima X and an ordinary abelian group A, the classical cohomology group $H^n(X, A)$ is represented by the Eilenberg-MacLane space $K(A, n)$, arising as n-delooping of A. As established in section 2, ∞-topoi possess well-behaved internal homotopy groups, rendering them a natural setting for generalized homotopy theory and allowing for a straightforward generalization of Eilenberg–MacLane spaces. In view of the classical identification $H^n(X, A) \cong [X, K(A, n)] = \pi_0 \operatorname{map}(X, K(A, n))$, the natural strategy is to define ∞-cohomology of an ∞-topos via mapping spaces into internal Eilenberg-MacLane objects.

Definition 4.1.1 ([HTT, 7.2.2.1]). Let $n \geq 0$. An **Eilenberg-MacLane object** of degree n in an ∞-topos $\mathcal{X}$ is a pointed, n-truncated and $n-1$-connected object. Equivalently, an Eilenberg-MacLane object is a pointed object – i.e. a morphism $1_{\mathcal{X}} \to X$ in $\mathcal{X}$ – where $1_{\mathcal{X}}$ is terminal and the only non vanishing homotopy group of X is concentrated in degree n.
Let $\mathcal{X}_* \subset \operatorname{Fun}(\Delta^1, \mathcal{X})$ denote the full subcategory of pointed objects[30] and let $\mathcal{EM}_n(\mathcal{X}) \subset \mathcal{X}_*$ denote the full subcategory of Eilenberg-MacLane objects of degree n.

These objects are characterized as deloopings of homotopy groups, extending the classical construction. This relationship is best understood via the loop space functor.

Definition 4.1.2. Let $\mathcal{C}$ be an ∞-category with finite limits. For any pointed object $p \colon 1_{\mathcal{C}} \to x$, define the **loop space object** $\Omega\, x$ to be the pullback

$$\begin{tikzcd}
\Omega\, x \arrow[r] \arrow[d] & 1_{\mathcal{C}} \arrow[d, "p"] \\
1_{\mathcal{C}} \arrow[r, "p"] & x
\end{tikzcd}$$

The functoriality of pullbacks yields the **loop space functor**

$$\Omega_{\mathcal{C}} \colon \mathcal{C}_* \to \mathcal{C},\ x \mapsto \Omega\, x.$$

Observation 4.1.3. Notice, that the loop space Ωx for a pointed object $p \colon 1_{\mathcal{C}} \to x$ is precisely the first level of the Čech nerve $\check{C}(p)_\bullet \in \mathcal{C}^{\Delta^{\mathrm{op}+}}$ i.e. $\check{C}(p)_1 \simeq \Omega x$. Thus, every loops space in $\mathcal{C}$ admits a natural interpretation as group object and Ω upgrades to a functor $\Omega_{\mathcal{C}} \colon \mathcal{C}_* \to \operatorname{Grp}_{\mathbb{E}_1}(\mathcal{C})$.

For an ordinary monoid the neutral element is taken as canonical choice of a base point. Thus, any group object in Set is pointed. Likewise, any group object of an ∞-category admits a natural interpretation as pointed object[31]. This yields the following fact.

Lemma 4.1.4 ([HTT, 7.2.2.10][GGN15]). *Let $\mathcal{C}$ be an ∞-category with finite products. Then, the forgetful functor $\mathcal{C}_* \xrightarrow{\mathrm{ev}_1} \mathcal{C}$ induces an equivalence* $\operatorname{Grp}_{\mathbb{E}_1}(\mathcal{C}_*) \simeq \operatorname{Grp}_{\mathbb{E}_1}(\mathcal{C})$.

This identification allows us to treat group objects independently of their pointing. In this light, the following recognition theorem of Lurie, which is stated here without proof, establishes the existence of deloopings for ∞-group objects.

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[30] i.e. $\mathcal{X}_*$ is the full subcategory of $\mathcal{X}^{\Delta^1}$ spanned by arrows of the form $1_{\mathcal{X}} \to X$, where $1_{\mathcal{X}}$ is a terminal object. Moreover, $\mathcal{X}_* \simeq \mathcal{X}^{1_{\mathcal{X}}/}$, since $1_{\mathcal{X}}$ is unique up to contractible choice of $\mathcal{X}$ (see [HTT, 7.2.2.8].

[31] Indeed, the forgetful functor $\operatorname{Grp}_{\mathbb{E}_1}(\mathcal{C}) \to \mathcal{C},\ M \mapsto M_1$ factors through $\mathcal{C}_*$. See [GGN15] for details.

Theorem 4.1.5 ([HTT, 7.2.2.11])**.** *For an ∞-topos $\mathcal{X}$ consider the composition*

$$p : \mathcal{X}^{\Delta^1} \xrightarrow{\check{C}} \mathcal{X}^{\Delta^{\mathrm{op}}_+} \xrightarrow{\text{forget}} \mathcal{X}^{\Delta^{\mathrm{op}}}$$

which maps a morphism f in $\mathcal{X}$ to the groupoid object underlying $\check{\mathrm{C}}(f)$.

1) *The restriction $p|_{\mathcal{X}_*}$ is equivalent to the loop space functor Ω of Obs. 4.1.3.*

2) *Let $\mathrm{Conn}\mathcal{X}_*$ be the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{X})$ of pointed connected objects of $\mathcal{X}$. Restricting p to $\mathrm{Conn}\mathcal{X}_*$ yields the equivalence*

$$p|_{\mathrm{Conn}\mathcal{X}_*} : \mathrm{Conn}\mathcal{X}_* \xrightarrow{\simeq} \mathrm{Grp}_{\mathbb{E}_1}(\mathcal{X})$$

Moreover, $p|_{\mathrm{Conn}\mathcal{X}_*}$ is equivalent to the loops space functor .

3) *Restricting p to Eilenberg-MacLane objects of degree $n \geq 1$ yields the equivalence*

$$p|_{\mathcal{EM}_n(\mathcal{X})} \colon \mathcal{EM}_n(\mathcal{X}) \xrightarrow{\simeq} \mathrm{Grp}_{\mathbb{E}_1}(\mathcal{EM}_{n-1}(\mathcal{X}))$$

where $\mathrm{Grp}_{\mathbb{E}_1}(\mathcal{EM}_{n-1}(\mathcal{X})) \subset \mathrm{Grp}_{\mathbb{E}_1}(\mathcal{X}_*)$ is a full subcategory of $\mathrm{Grp}_{\mathbb{E}_1}(\mathcal{X})$ by Lem. 4.1.4.

Definition 4.1.6 (Delooping)**.** The essentially unique inverse to the functor $\Omega \simeq p|_{\mathrm{Conn}\mathcal{X}_*}$ is called the **delooping functor** and is denoted by $\mathbf{B} : \mathrm{Grp}_{\mathbb{E}_1}(\mathcal{X}) \to \mathrm{Conn}\mathcal{X}_*$.

Observation 4.1.7. Let $g^* \colon \mathcal{X} \to \mathcal{Y}$ be a morphism in $\mathcal{LT}\mathrm{op}$. As such, it preserves finite limits and therefore $g^* \circ \Omega_{\mathcal{X}} \simeq \Omega_{\mathcal{Y}} \circ g^*$ as checked pointwise. Note, that both compositions restrict to functors $\mathrm{Conn}\mathcal{X}_* \to \mathrm{Grp}_{\mathbb{E}_1}(\mathcal{Y})$, since g^* preserves groupoid objects. In consequence, g^* commutes with $\mathbf{B}$.

Motivated by the fact, that classical cohomology is representable by Eilenberg-MacLane spaces, the ∞-cohomology groups are defined as mapping spaces of n-th deloopings.

Definition 4.1.8 (∞-Cohomology)**.** For any two objects X, A of an ∞-topos $\mathcal{X}$ one sets

$$H^0(X, A) := \pi_0 \operatorname{map}_{\mathcal{X}}(X, A)$$

If A admits a n-fold delooping $\mathbf{B}^n A$ for $n \geq 1$, define

$$H^n(X, A) := \pi_0 \operatorname{map}_{\mathcal{X}}(X, \mathbf{B}^n A)$$

Denote $H^*(\mathcal{X}, A) := H^*(1_{\mathcal{X}}, A)$.

To obtain a full cohomology theory with coefficients in any discrete group object of $\mathcal{X}$, it suffices to show that every such object admits a delooping. For this purpose, it is convenient to introduce the following abbreviations.

Notation 4.1.9. Let $\mathcal{G}\mathrm{rp}(\mathcal{C})$ and $\mathcal{A}\mathrm{b}(\mathcal{C})$ denote the discrete and discrete commutative group objects of an ∞-category $\mathcal{C}$. Hence, $\mathcal{G}\mathrm{rp}(\mathcal{C}) \simeq \mathrm{Grp}_{\mathbb{E}_1}(\tau_{\leq 0}\mathcal{C})$ and $\mathcal{A}\mathrm{b}(\mathcal{C}) \simeq \mathrm{Grp}_{\mathbb{E}_\infty}(\tau_{\leq 0}\mathcal{C})$. Thus, $\mathcal{G}\mathrm{rp}(\mathcal{C})$ and $\mathcal{A}\mathrm{b}(\mathcal{C})$ are the nerves of the categories of (abelian) group objects of the 1-category $\mathrm{Disc}(\mathcal{C})$ of Not. 2.6.2.

Similar to the classical case, Eilenberg–MacLane objects are uniquely determined by their homotopy groups, as the latter may be described via loops spaces.

Remark 4.1.10. For a pointed object $1_{\mathcal{X}} \to X$ in an ∞-topos $\mathcal{X}$ the homotopy groups of the loops space ΩX behave as expected. By applying the reasoning of [HTT, 6.5.1.5] to the morphisms $\Omega X \to 1_{\mathcal{X}} \to X$ one obtains a long exact sequence of homotopy groups in $\mathrm{Disc}(\mathcal{X}_{/\Omega X})$, which exhibits that $\pi_{n+1}(X)$ and $\pi_n \Omega X$ coincide in $\mathrm{Disc}(\mathcal{X}_{/\Omega X})$ for all $n \geq 0$. In particular, $\pi_n X \simeq \pi_0 \Omega^n X$ for all $n \geq 0$.

Proposition 4.1.11 ([HTT, 7.2.2.12])**.** *Let $\mathcal{X}$ be an ∞-topos. For any $n \geq 0$, let*

$$\pi_n := \tau_{\leq 0} \circ \Omega^n \colon \mathcal{X}_* \to \tau_{\leq 0}\mathcal{X}$$

denote the functor that assigns each $(p\colon 1_{\mathcal{X}} \to X) \in \mathcal{X}_*$ the n-th homotopy group $p^*\pi_n(X)$. Restricting π_n to Eilenberg-MacLane objects of degree n, provides the equivalences

- *0)* $\pi_0|_{\mathcal{EM}_0(\mathcal{X})} \colon \mathcal{EM}_0(\mathcal{X}) \xrightarrow{\simeq} \tau_{\leq 0}(\mathcal{X})_*$ *for* $n = 0$.
- *1)* $\pi_1|_{\mathcal{EM}_1(\mathcal{X})} \colon \mathcal{EM}_1(\mathcal{X}) \xrightarrow{\simeq} \mathcal{G}\mathrm{rp}(\mathcal{X})$ *for* $n = 1$. *Thus,*
- *n)* $\pi_n|_{\mathcal{ME}_n}(\mathcal{X}) \colon \mathcal{EM}_n \xrightarrow{\simeq} \mathcal{A}\mathrm{b}(\mathcal{X})$ *for* $n \geq 2$.

This formalizes the standard behaviour: ∞-group objects admit a single delooping, whereas abelian ∞-group objects may be delooped indefinitely.

Proof. The case $n = 0$ is trivial, since $\mathcal{EM}_0(\mathcal{X}) = \tau_{\leq 0}(\mathcal{X})_*$ by Definition and $\pi_0|_{\mathcal{EM}_0} \simeq \mathrm{id}$.
By Obs. 4.1.3 the functor π_n lands in $\mathcal{G}\mathrm{rp}(\mathcal{X})$ for any $n \geq 1$. Now, the case $n = 1$ coincides precisely with the statement of Thm. 4.1.5.2) restricted to 1-truncated objects. To prove the general case, observe that the usual Eckmann-Hilton argument yields
$\mathcal{G}\mathrm{rp}(\mathcal{G}\mathrm{rp}(\mathcal{X})) \cong \mathcal{A}\mathrm{b}(\mathcal{X})$ and $\mathcal{G}\mathrm{rp}(\mathcal{A}\mathrm{b}(\mathcal{X})) \cong \mathcal{A}\mathrm{b}(\mathcal{X})$, when we identify these ∞-categories with the (abelian) group objects in the 1-category $\mathrm{Disc}(\mathcal{X})$. Invoking an induction on $n \geq 1$, we deduce

$$\begin{aligned} \tau_{\leq 0}\Omega^{n+1}\mathcal{EM}_{n+1}(\mathcal{X}) &\overset{4.1.5}{\simeq} \tau_{\leq 0}\Omega^n \,\mathrm{Grp}_{\mathbb{E}_1}(\mathcal{ME}_n(\mathcal{X})) \simeq \mathcal{G}\mathrm{rp}\,\tau_{\leq 0}\Omega^n \mathcal{ME}_n(\mathcal{X}) \\ &\overset{\text{induction}}{\simeq} \begin{Bmatrix} \mathcal{G}\mathrm{rp}(\mathcal{G}\mathrm{rp}(\mathcal{X})) & \text{if } n = 1 \\ \mathcal{G}\mathrm{rp}(\mathcal{A}\mathrm{b}(\mathcal{X})) & \text{if } n \geq 2 \end{Bmatrix} \cong \mathcal{A}\mathrm{b}(\mathcal{X}). \end{aligned}$$ ☐

Example 4.1.12 (Classical Cohomology, [HTT, 7.2.2.17])**.** Let (C, τ) be a 1-site with a terminal object $X \in \mathcal{C}$, mimicking the usual setup, where C is the small site of a space or scheme X. Let $F \in \mathrm{Shv}_\tau(C; \mathrm{Ab})$ be a sheaf of abelian groups. As explained in § 2.11, when viewing F as sheaf with values in chain complexes, it may be regarded as object

$$F[0] \in \mathrm{Shv}_\tau(C; \mathrm{Ch}_+(\mathrm{Ab})) \cong \mathrm{Shv}_\tau(C; s\mathrm{Ab}) \subset \mathrm{Shv}_\tau(C)$$

via the Dold-Kan correspondence[32]. Since $F[0] \in \mathcal{A}\mathrm{b}(\mathrm{Shv}_\tau(\mathcal{C}))$, each cohomology object $\pi_0\,\mathrm{map}(X, \mathbf{B}^n F[0])$,

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[32] c.f. [Wei94, 8.4.1]

for $n \geq 0$, is an abelian group. Indeed, the canonical equivalence

$$\mathrm{Shv}_\tau(C, s\mathrm{Ab}) \cong \mathrm{Shv}_\tau(C; \mathrm{Ch}_+(\mathrm{Ab})) \cong \mathrm{Ch}_+(\mathrm{Shv}_\tau(C; \mathrm{Ab})).$$

is given by applying the normalized chain complex functor $N^\bullet$of [Wei94, 8.3.6] pointwise for each sheaf. The first isomorphism sends the terminal object $X \colon C^{\mathrm{op}} \to s\mathrm{Ab} \subset \mathrm{Ani}$, $U \mapsto \Delta^0$ of $\mathrm{Shv}_\tau(C)$ to

$$\mathbb{Z}[X] \colon C \to \mathrm{Ch}_+(\mathrm{Ab}),\ U \mapsto N^\bullet(\Delta^0) = \mathbb{Z}[0]$$

This is then regarded as chain complex of sheaves under the second isomorphism. Likewise, $\mathbf{B}^n F \in$ is sent to to $F[n]$. Moreover, the equivalence above yields the identification

$$\begin{aligned} &\pi_0 \operatorname{map}_{\mathrm{Shv}_\tau(C; s\mathrm{Ab})}(X, \mathbf{B}^n F) \simeq \pi_0 \operatorname{map}_{\mathrm{Shv}_\tau(C; \mathrm{Ch}_+(\mathrm{Ab}))}(\mathbb{Z}[X]; F[n]) \\ &= \mathrm{Hom}_{h(\mathrm{Ch}_+(\mathrm{Shv}_\tau(C;\mathrm{Ab})))}(\mathbb{Z}[X], F[n]) \cong H^0 \mathrm{Hom}_{D(C)}(\mathbb{Z}[X], F[n]) \overset{[\text{Wei94, 10.7}]}{\cong} H^n \mathrm{Hom}(\mathbb{Z}[X], F). \end{aligned}$$

where $D(C)$ is the derived category of chain complexes in $\mathrm{Shv}_\tau(C; \mathrm{Ab})$ and $\mathrm{Hom}_{D(C)}$ denotes the associated chain complex. Thinking of $\mathbb{Z}[X]$ as the chain complex given by the image of X under Yoneda embedding[33] into $\mathrm{Shv}_\tau(C; \mathrm{Ab})$ centered at 0, it follows that

$$H^n \mathrm{Hom}(\mathbb{Z}[X], F) \cong H^n R\Gamma(F) \overset{[\text{Wei94, 10.6.8}]}{\cong} H^n(X, F).$$

by the Yoneda Lemma.

4.2 Principal ∞-Bundles

Classically, homotopy classes of maps into the delooping $\mathbf{B}G$ of an ordinary group G classify principal G-bundles in the category of topological spaces. This section establishes that such a correspondence persists within the broader framework of ∞-topoi. Equivalently, the first ∞-cohomology groups introduced in the preceding sections admit a unique characterization in terms of equivalence classes of general principal bundles internal to the surrounding ∞-topos, thereby endowing them with a rich geometric structure. Following the outline of [NSS14], only a brief overview of the theory is provided, with proofs omitted.

Definition 4.2.1 ([NSS14, Def.3.2]). Let $\mathcal{X}$ be an ∞-topos and $G \in \mathrm{Grp}_{\mathbb{E}_1}(\mathcal{X})$ a group object. A **G-action** on an object $P \in \mathcal{X}$ is a groupoid object $P//G_\bullet \in \mathcal{X}^\Delta$ of the form

$$\cdots \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} P \times G \times G \substack{\rightarrow\\\rightarrow\\\rightarrow} P \times G \underset{d_1}{\overset{d_0}{\rightrightarrows}} P$$

where $d_1 : P \times G \to P$ is the projection induced by $P \to 1_{\mathcal{X}}$ and $P \times G^n \to G^n$ assemble into a morphism of groupoid objects

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[33] This is precisely the sheaf

$$y_{\mathbb{Z}}(X) \colon C^{\mathrm{op}} \to \mathrm{Ab},\ U \mapsto \mathbb{Z}\,\mathrm{Hom}_C(U, X) = \mathbb{Z}.$$

$$\begin{tikzcd}
\cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & P \times G \times G \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d] & P \times G \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & P \arrow[d] \\
\cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & G \times G \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & G \arrow[r, shift left] \arrow[r, shift right] & 1_{\mathcal{X}}
\end{tikzcd}$$

where the bottom simplicial object exhibits G as the groupoid $1_{\mathcal{X}}//G_\bullet$. Write

$$P//G := \operatorname{colim}(P \times G^{\times^\bullet}) \quad \in \mathcal{X}$$

for the associated ∞-colimit of $P//G_\bullet$. and let $G\mathrm{Action}(\mathcal{X}) \subset \mathrm{Grpd}(\mathcal{X})_{/(1_{\mathcal{X}}//G)}$ denote sub-∞-category spanned by G-actions.

Finally, a morphism $P \to X$ together with a G-action $(P//G)_\bullet$ is a **principal G-bundle over X** if $X \simeq P//G$. Define $G\mathrm{Bun}_{\mathcal{X}}(X) := G\mathrm{Action}(\mathcal{X}) \times_{\mathcal{X}} \{X\}$ and call this ∞-category the category of principal G-bundles.
In case $X \simeq 1_{\mathcal{X}}$ is a terminal object, abbreviate $G\mathrm{Bun}_{\mathcal{X}} := G\,\mathrm{Bun}_{\mathcal{X}}(1_{\mathcal{X}})$.

Observation 4.2.2. Morphism in $\mathcal{LT}$op preserves group actions and principal ∞-bundles, as they preserves colimits and finite limits.

Every map into $\mathbf{B}G$ canonically gives rise to a principal G-bundle.

Observation 4.2.3 ([NSS14, 3.8])**.** For a $G \in \mathrm{Grp}_{\mathbb{E}_1}(\mathcal{X})$ the delooping $\mathbf{B}G$ is canonically pointed by Thm. 4.1.5. Now, for any map $X \to \mathbf{B}G$ in $\mathcal{X}$ the fiber $P := X \times_{\mathbf{B}G} 1_{\mathcal{X}}$ carries the structure of a principal G-bundle, since pulling back along the morphisms of the groupoid $1_{\mathcal{X}}//G_\bullet$ provides the diagram

$$\begin{tikzcd}
\cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & P \times G \times G \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d] & P \times G \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & P \arrow[r] \arrow[d] & X \arrow[d] \\
\cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & G \times G \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & G \arrow[r, shift left] \arrow[r, shift right] & 1_{\mathcal{X}} \arrow[r] & \mathbf{B}G
\end{tikzcd}$$

where every square is a pullback square. Thus, the upper row is a groupoid and equips P with the canonical structure of a principal G-action.

Example 4.2.4. Let $G \in \mathrm{Grp}_{\mathbb{E}_1}(\mathcal{X})$. Every $X \in \mathcal{X}$ admits a trivial principal G-bundle $X \times G \to X$ arising from the canonical map $X \to 1_{\mathcal{X}} \to \mathbf{B}G$ via Obs. 4.2.3.

Moreover, Def. 4.2.1 directly extends the classical notion of principal bundles.

Example 4.2.5. Classically, an action $G \curvearrowright E$ of a topological group on a topological space E gives rise to a principal bundle $p\colon E \to E/G$ precisely if p admits a local sections and the shear map

$$E \times G \to E \times E, (e, g) \to (e, eg)$$

is an homeomorphism (cf. [Die10, 14.1.4]).
Let (C, τ) be a 1-presite with finite products and let $G \in C$ be a group object. A principal

G-action on an object $P \in C$ is simply a map $a\colon P \times G \to P$ such that

$$P \times G \xrightarrow{a \times pr_1} P \times P$$

is an isomorphism. Note we do not require that the quotient of this action

$$P/G := \operatorname{coeq}(P \times G \overset{a}{\underset{pr_P}{\rightrightarrows}} P).$$

exists. In the category of schemes for example, this is rarely the case (c.f. [Mil17]). If it does exist, then $P \to P/G$ is said to be a principal G-bundle, if there exists a τ-covering $\coprod_i U_i \xrightarrow{P} /G$, such that $P \times_{P/G} U_i \to U_i$ is the trivial G-bundle.
In general, every principal G-action on P in C induces is a principal $Ly(G)$-bundle $L_\tau y(P) \in \mathrm{Shv}_\tau(\mathcal{C})$ by setting $d_0 = L_\tau y(a)$ and $d_1 = pr_1$ and extending this definition to an simplicial object. If the quotient $X := P//G \in C$ does exist, then $L_\tau y(P)//L_\tau y(G) \simeq L_\tau y(X)$, since the Yoneda embedding y is fully faithful and L_τ is preserves colimits and finite limits.

It might seem suspicious that we did not require a shear-map or locality condition on the action of a principal ∞-bundle, as commonly done for principal bundles in topological space. This is simply not necessary in this context of ∞-topoi. For once, Prop. 4.2.6 below states that every principal ∞-bundle in the sense above already is locally trivial. On the other hand, for every principal ∞-bundle one has $P//G_\bullet \simeq \check{\mathrm{C}}(P \to P//G)$ since groupoids are effective and thus

$$P \times G = P//G_1 \simeq \check{\mathrm{C}}(P \to P//G)_1 = P \times_X P.$$

Proposition 4.2.6 ([NSS14, Prop.3.11])**.** *Every principal ∞-bundle is locally trivial i.e. for every $G \in \mathrm{Grp}_{\mathbb{E}_1}(\mathcal{X})$ and principal G-bundle P over X, there is an effective epimorphism $U \twoheadrightarrow X$, such that $P \times_X U$ is the trivial bundle of Ex. 4.2.4.*

Not only principal ∞-bundles behave as expected, but so do their morphisms.

Lemma 4.2.7. *Every map between principal ∞-bundles over a fixed base space $X \in \mathcal{X}$ is an equivalence.*

As taking fibres is functorial, the assignment of Obs. 4.2.3:

$$(f\colon X \to \mathbf{B}G) \mapsto (X \times_{\mathbf{B}G} 1_{\mathcal{X}})//G_\bullet \in G\mathrm{Bun}_{\mathcal{X}}(X)$$

extends to a functor

$$\psi\colon \operatorname{map}_{\mathcal{X}}(X, \mathbf{B}G) \to G\mathrm{Bun}_{\mathcal{X}}(X).$$

Nikolaus, Schreiber & Stevenson showed, that ψ yields the following classification.

Theorem 4.2.8 ([NSS14, 3.17])**.** *There is an equivalence of functors*

$$\mathrm{GBun}_{\mathcal{X}}(-) \simeq \operatorname{map}_{\mathcal{X}}(-, \mathbf{B}G)$$

for any ∞-topos $\mathcal{X}$ and any $G \in \mathrm{Grp}_{\mathbb{E}_1}(\mathcal{X})$).

Corollary 4.2.9. *Postcomposing with* $\pi_0 : \mathrm{An} \to \mathrm{Set}$*, the equivalence above yields*

$$\pi_0 GBun_{\mathcal{X}}(-) \simeq H^1(-, \mathbf{B}G).$$

Corollary 4.2.10. *By Theorem 4.2.8, principal bundles with constant group are classified by the shape of* $\mathcal{X}$*. That is, for every* $G \in \mathrm{Grp}_{\mathbb{E}_1}(\mathrm{Ani})$ *there is an equivalence functorial in* $X \in \mathcal{X}$

$$\underline{G}Bun_{\mathcal{X}}(X) \simeq \Pi_\infty^{\mathcal{X}}(X)(\mathbf{B}G).$$

4.3 ∞-Fiber Bundles

Recall from any standard textbook on classical topology (e.g. [Die10]) that, for a topological space, V-fiber bundles directly translate to $\mathrm{Aut}(V)$-principal bundles. This correspondence extends naturally to ∞-topoi. Following [NSS14, §4.1], fiber bundles internal to an ∞-topos can be related to principal bundles, yielding a classification via Theorem 4.2.8. As in the classical case, the notion of a V-fiber bundle over an object $X \in \mathcal{X}$ stems from the trivial bundle $V \times X \to X$ in the following way.

Definition 4.3.1 (Fiber Bundle)**.** Let $V, X \in \mathcal{X}$ be objects in a ∞-topos $\mathcal{X}$. A morphism $p\colon E \to X$ is a V-fiber ∞-bundle, if $p\colon E \to X$ is locally isomorphic to the trivial V-bundle i.e. there is an effective epimorphism $U \to X$ giving rise to a pullback diagram

$$\begin{tikzcd}
U \times V \arrow[r] \arrow[d] & E \arrow[d] \\
U \arrow[r] & X
\end{tikzcd}$$

Let $V\mathrm{Bun}(X) \subset \mathcal{X}_{/X}$ denote the full subcategory of V-bundles over X.

In order to define the automorphism groups internal to $\mathcal{X}$, another standard property of ∞-topoi is needed. As shown by Rezk, ∞-topoi can be characterized by the existence of object classifiers for object relatively compact with respect to a sufficiently large cardinal. Extending the notion of κ-compact objects[34] this is defined as follows.

Definition 4.3.2 ([HTT, 6.1.6.4])**.** A morphism $X \to Y$ is **relatively κ-compact** if for every map $Y' \to Y$ with Y' κ-compact, the pullback $Y' \times_Y X$ is again κ-compact.

All κ-compact anima give rise to relatively κ-compact morphisms.

Example 4.3.3. For κ-compact $X, Y \in \mathrm{Ani}$, their product $X \times Y$ is κ-compact, since $\mathrm{map}_{\mathrm{Ani}}(X \times Y, -) \simeq \mathrm{map}_{\mathrm{Ani}}(X, \mathrm{map}_{\mathrm{Ani}}(Y, -))$ preserves filtered colimits. Thus, the canonical map $X \to *$ is relatively κ-compact.

The following characterisation – stated here without proof – is originally due to C.Rezk.

Theorem 4.3.4 (Characterisation of ∞-Topoi, [HTT, 6.1.6.8])**.** *A presentable ∞-category $\mathcal{X}$ is an ∞-topos if and only if*

1) Colimits in $\mathcal{X}$ are universal.

---

[34] Recall, that $X \in \mathcal{X}$ is κ-compact for a cardinal κ if $\mathrm{map}\,\mathcal{X}(X, -)$ preserves κ-filtered colimits.

2) For any sufficiently large cardinals κ, there is an object classifier for relatively κ-compact morphisms i.e. there is a relatively κ-compact morphism $\widehat{\mathrm{Obj}}_\kappa \to \mathrm{Obj}_\kappa$, such that every κ-compact morphism $Y \to X$ is equivalent to the pullback of $\widehat{\mathrm{Obj}}_\kappa \to \mathrm{Obj}_\kappa$ a morphism $X \to \mathrm{Obj}_\kappa$, which is unique up to equivalence.
Equivalently, $\widehat{\mathrm{Obj}}_\kappa \to \mathrm{Obj}_\kappa$ is a classifier for relatively κ-compact morphisms, if pulling back this morphism yields an equivalence

$$(\mathcal{X}_{/_\kappa X})^{\simeq}\, \mathrm{map}_{\mathcal{X}}(X, \mathrm{Obj}_\kappa)$$

for every $X \in X$. Here $\mathcal{X}_{/_\kappa X}$ denotes the full subcategory of κ-compact objects over X and $(\mathcal{X}_{/_\kappa X})^{\simeq}$ is the groupoid core of this category.

By the above, ∞-topoi provide a sufficient supply of object classifiers, allowing for the following definition.

Definition 4.3.5 (Internal Automorphism Group, [NSS14, 4.9])**.** Let V be an object of $\mathcal{X}$ and let κ be a sufficiently large cardinal, such that V is κ-compact and there exists an object classifier for κ-compact objects. By Thm. 4.3.4, there is a pullback diagram

$$\begin{tikzcd}
V \arrow[r] \arrow[d] & \widehat{\mathrm{Obj}}_\kappa \arrow[d] \\
1_{\mathcal{X}} \arrow[r, "\vdash_V"] & \mathrm{Obj}_\kappa
\end{tikzcd}$$

By Prop. 2.3.16, the morphism $\vdash_V$ admits a factorization into an effective epimorphism followed by an monomorphism

$$\vdash_V \colon 1_{\mathcal{X}} \overset{e}{\twoheadrightarrow} \mathrm{im}(\vdash_V) \overset{m}{\hookrightarrow} \mathrm{Obj}_\kappa$$

where $\mathrm{im}(\vdash_V) \in \mathcal{X}$ is the unique intermediate object of this factorization. The map $e \colon 1_{\mathcal{X}} \twoheadrightarrow \mathrm{im}(\vdash_V)$ exhibits $\mathrm{im}(\vdash_V)$ as a delooping of $\check{\mathrm{C}}(e)_\bullet \in \mathrm{Grp}_{\mathbb{E}_1}(\mathcal{X})$. Thus, setting

$$\mathrm{Aut}(V) := \Omega\ \mathrm{im}(\vdash_V)$$

produces a well defined ∞-group of $\mathcal{X}$.

By identifying each V-fiber bundle with an principal $\mathrm{Aut}(V)$-bundle, Nikolaus, Schreiber & Stevenson have established the following classification, which we simply state as a fact.

Theorem 4.3.6 (Classification of Fiber ∞-Bundles, [NSS14, 4.11])**.** *For every ∞-topos $\mathcal{X}$ and any $V, X \in \mathcal{X}$, there is an equivalence*

$$V\mathrm{Bun}(X) \simeq \mathrm{map}_{\mathcal{X}}(X, \mathbf{B}\,\mathrm{Aut}(V))$$

where $V\mathrm{Bun}(X)$ is the ∞-category of V-fiber bundles over X and $\mathrm{Aut}(V)$ is the automorphism group of V internal to $\mathcal{X}$.

Although the definition of the automorphism groups seems rather technical, the Classification Theorem 4.3.6 shows, that Def. 4.3.5 produces the familiar group anima of homotopy equivalences, when applied in the context of anima in the following way.

Example 4.3.7. Let V be a simplicial set. May showed, that $\mathrm{map}_{\mathrm{Ani}}(V, V)$ is a monoid object in sSet (cf.[May92, 6.12]) and defines the automorphism set $A(V) \subset \mathrm{map}_{\mathrm{Ani}}(V, V)$ as the sub-simplicial set, which contains only the invertible elements of $\mathrm{map}_{\mathrm{Ani}}(V, V)$ in each level (see [May92, §19]). May also showed, that $A(K)$ is a group object in sSet, thus, in particular in anima (cf.[May92, 17.1]). Moreover, May defines an analogue of ∞-bundles in sSet with fiber V in [May92, 11.8] and shows, that the isomorphism classes of such V-fiber bundles over an $B \in \mathrm{Ani}$ are in bijection with the homotopy classes of maps $B \to \mathbf{B}A(V)$ (see [May92, 21.13])[35] Combining these facts with Thm. 4.3.6 implies that in $h\mathrm{Ani}$

$$[B, \mathbf{B}A(V)] \cong V\mathrm{Bun}(B)_{/\simeq} \cong [B, \mathbf{B}\,\mathrm{Aut}(V)].$$

Here, $[-,-]$ are the Hom-sets in $h\mathrm{Ani}$. An application of the Yoneda Lemma, thus yields

$$\mathbf{B}A(V) \simeq \mathbf{B}\,\mathrm{Aut}(V).$$

Note as well, that for a $V \in \mathrm{Ani}$ the 0-level $A(V)_0$ of the Kan complex $A(V)$ coincides with the group of homotopy equivalences $V \to V$.

4.4 Automorphism Groups of Locally Constant Sheaves

The preceding example shows that automorphism groups in Ani agree with their classical counterparts and therefore admit explicit descriptions. For later applications, in particular Thm. 4.6.1, it is essential to verify that the automorphism group of a (locally) constant object internal to an ∞-topos $\mathcal{X}$ is itself (locally) constant. This ensures that the explicit descriptions available in Ani carry over to $\mathcal{X}$. The present subsection is devoted to establishing this result.

The following statement is a first step towards this goal, as it provides a direct way to determine κ-compact morphism in Ani. The Lemma and its proof are inspired by [nLa25d].

Lemma 4.4.1. *A morphism $X \to Y$ in* Ani *is relatively κ-compact if and only if each fiber $X_y = X \times_Y \{y\}$ are κ-compact for any $y \in Y$.*
In particular, if $X \to Y$ is a map in Ani *and Y is κ-compact, then so is X.*

Proof. The "only if"-direction is obvious, as every $\{y\} \cong \Delta^0$ is compact for every cardinal.
For the other implication, let $Z \in \mathrm{Ani}$ be κ-compact and $f\colon Z \to Y$ a map in Ani. Recall that $Z \cong \mathrm{colim}_{z\in Z}\{z\}$. Since colimits in Ani are universal, the associated the pullback diagram reads as

$$\begin{tikzcd}
\underset{z\in Z}{\mathrm{colim}} X_{f(z)} \arrow[r, "\simeq"] \arrow[d] & X \times_Y Z \arrow[r] \arrow[d] & X \arrow[d] \\
\underset{z\in Z}{\mathrm{colim}} \arrow[r, "\simeq"] & Z \arrow[r] & Y
\end{tikzcd}$$

By [HTT, 5.3.4.15], the colimit of κ-compact anima over a κ-compact indexing category is again a κ-compact anima. Thus, $X \times_Y Z$ is κ-compact, since each $X_{f(z)}$ is.

---

[35] When looking into [May92, 21.13] the interested reader should note that May uses the non-standard notation $\bar{W}$ for the classical delooping functor as defined [Wei94, §8].

The last statement follows if one replaces Z by Y in the previous paragraph. ☐

Lemma 4.4.2. *Let $\mathcal{X}$ be an ∞-topos and $1_{\mathcal{X}}$ a terminal object. For a κ-compact $X \in \mathrm{Ani}$ the unique map $\underline{\Delta}_{\mathcal{X}}(X) \to 1_{\mathcal{X}}$ is relatively κ-compact.*

Proof. The proof is similar to Ex. 4.3.3. Let $Y \in \mathcal{X}$ be a κ-compact object. Then,

$$\begin{aligned} \mathrm{map}_{\mathcal{X}}(\underline{\Delta}_{\mathcal{X}}(X) \times Y, -) &\overset{\times \dashv \underline{\mathrm{Hom}}}{\simeq} \mathrm{map}_{\mathcal{X}}(\underline{\Delta}_{\mathcal{X}}(X), \underline{\mathrm{Hom}}_{\mathcal{X}}(Y, -)) \\ &\overset{\Delta \dashv \Gamma}{\simeq} \mathrm{map}_{\mathrm{Ani}}(X, \Gamma \underline{\mathrm{Hom}}_{\mathcal{X}}(Y, -)) \simeq \mathrm{map}_{\mathrm{Ani}}(X, \mathrm{map}_{\mathcal{X}}(1_{\mathcal{X}}, \underline{\mathrm{Hom}}_{\mathcal{X}}(Y, -)) \\ &\simeq \mathrm{map}_{\mathrm{Ani}}(X, \mathrm{map}_{\mathcal{X}}(Y, -)). \end{aligned}$$

This functor preserves κ-filtered colimits, thus $\underline{\Delta}_{\mathcal{X}}(X) \times Y$ is κ-compact. ☐

Corollary 4.4.3. *Let $\mathcal{X}$ be a ∞-topos and let κ be a sufficiently large cardinal, such that $\mathcal{X}$ contains an object classifying relatively κ-compact morphisms. Then, the geometric morphism $\underline{\Delta}\colon \mathrm{Ani} \to \mathcal{X}$ preserves relatively κ-compact morphisms.*

Proof. Let $X \to Y$ be a relatively κ-compact morphism in Ani. By Lem. 4.4.1, the fibre X_y is a κ-compact anima for each $y \in Y$ and we may write $X \simeq \mathrm{colim}_{y \in Y} X_y$. Let $\widehat{\mathrm{Obj}}_{\kappa}^{\mathcal{X}} \to \mathrm{Obj}_{\kappa}^{\mathcal{X}}$ be the classifying object for κ-compact morphisms in $\mathcal{X}$. By Lem. 4.4.2 there is a pullback square for every $y \in Y$

$$\begin{tikzcd} \underline{X_y} \arrow[r] \arrow[d] & \widehat{\mathrm{Obj}}_{\kappa}^{\mathcal{X}} \arrow[d] \\ 1_{\mathcal{X}} \simeq \underline{\{y\}} \arrow[r] & \mathrm{Obj}_{\kappa}^{\mathcal{X}} \end{tikzcd}$$

Thus the following square, obtained by the universal property of the colimit, is a pullback, because colimits are universal in $\mathcal{X}$

$$\begin{tikzcd} \underline{X} \arrow[r, "\simeq"] \arrow[d] & \underset{y \in Y}{\mathrm{colim}} \underline{X_y} \arrow[r] \arrow[d] & \widehat{\mathrm{Obj}}_{\kappa}^{\mathcal{X}} \arrow[d] \\ \underline{Y} \arrow[r, "\simeq"] & \underset{y \in Y}{\mathrm{colim}} \underline{\{y\}} \arrow[r] & \mathrm{Obj}_{\kappa}^{\mathcal{X}} \end{tikzcd}$$

☐

Cor. 4.4.3 admits an analogue for étale morphisms of ∞-topoi:

Lemma 4.4.4. *Let κ be a regular cardinal. Let $\mathcal{X}$ be an ∞-topos and $U \in \mathcal{X}$. The geometric morphism $t_U^*\colon \mathcal{X} \to \mathcal{X}_{/U}, X \mapsto X \times_{1_{\mathcal{X}}} U$ preserves relatively κ-compact morphisms.*

Proof. Let $X \to Y$ be a relatively κ-compact morphism in $\mathcal{X}$. Let $K \in \mathcal{X}_{/U}$ be a κ-compact object. Given a morphism $K \to Y$ in $\mathcal{X}_{/U}$, denote the associated pullback by $P := K \times_{t_U^*(Y)} t_U^*(X)$. Applying the forgetful functor $\mathcal{X}_{/U} \to \mathcal{X}$ to the associated pullback diagram yields the following diagram in $\mathcal{X}$

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & X \times U \arrow[r] \arrow[d] & X \arrow[d] \\
K \arrow[r] & Y \times U \arrow[r] \arrow[d] & Y \arrow[d] \\
 & U \arrow[r] & 1_{\mathcal{X}}
\end{tikzcd}$$

By pullback pasting, every square in this diagram is a pullback square. Therefore, P is a κ-compact object in $\mathcal{X}$, since $X \to Y$ is relatively κ-compact. Let $W_{-}\colon I \to \mathcal{X}_{/U}$ be a κ-filtered diagram in $\mathcal{X}_{/U}$. The following equivalences, verify that P is κ-compact

$$\begin{aligned}
&\operatorname*{colim}_{i\in I} \operatorname{map}_{\mathcal{X}_{/U}}(P, W_i) \overset{\text{A.1.2}}{\simeq} \operatorname*{colim}_{i\in I}(\operatorname{map}_{\mathcal{X}}(P, W_i) \times_{\operatorname{map}_{\mathcal{X}}(P,U)} \{P \to U\}) \\
&\simeq (\operatorname*{colim}_{i\in I} \operatorname{map}_{\mathcal{X}}(P, W_i)) \times_{\operatorname{map}_{\mathcal{X}}(P,U)} \{P \to U\} \simeq \operatorname{map}_{\mathcal{X}}(P, \operatorname*{colim}_{i\in I} W_i) \times_{\operatorname{map}_{\mathcal{X}}(P,U)} \{P \to U\} \\
&\simeq \operatorname{map}_{\mathcal{X}_{/U}}(P, \operatorname*{colim}_{i\in I} W_i)
\end{aligned}$$

Here we have used, that colimits in Ani are universal. ☐

The following Proposition allows us to relate automorphisms groups of constant objects back to the usual (homotopy) automorphism groups of topological spaces.

Proposition 4.4.5. *Let $\mathcal{X}$ be an ∞-topos and let κ be a sufficiently large cardinal, such that $\mathcal{X}$ and* Ani *contain classifying objects for relatively κ-compact morphisms.*

1) For every κ-compact object $V \in \mathrm{Ani}$, the geometric morphism $\underline{\Delta}\colon \mathrm{Ani} \to \mathcal{X}, V \mapsto \underline{V}$ induces an equivalence

$$\mathbf{B}\operatorname{Aut}(\underline{V}) \simeq \underline{\mathbf{B}\operatorname{Aut}(V)}.$$

2) Let $U, V \in \mathcal{X}$ be objects and assume V is κ-compact. Let $t\colon U \to 1_{\mathcal{X}}$ the morphism to a terminal object . Then, the associated t^*: geometric morphism induces the following equivalence in $\mathcal{X}_{/U}$

$$\mathbf{B}\operatorname{Aut}(V \times U) \simeq \mathbf{B}\operatorname{Aut}(V) \times U.$$

In particular, there are commutative squares

$$\begin{tikzcd}
\mathrm{Ani} \arrow[r, "\underline{\Delta}"] \arrow[d, "\operatorname{Aut}(-)"] & \mathcal{X} \arrow[d, "\operatorname{Aut}(-)"] \\
\mathrm{Grp}_{\mathbb{E}_1}(\mathrm{Ani}) \arrow[r, "\underline{\Delta}"] & \mathrm{Grp}_{\mathbb{E}_1}(\mathcal{X})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{X} \arrow[r, "-\times U"] \arrow[d, "\operatorname{Aut}(-)"] & \mathcal{X}_{/U} \arrow[d, "\operatorname{Aut}(-)"] \\
\mathrm{Grp}_{\mathbb{E}_1}(\mathcal{X}) \arrow[r, "-\times U"] & \mathrm{Grp}_{\mathbb{E}_1}(\mathcal{X}_{/U})
\end{tikzcd}$$

Proof. We restrict ourselves to verifying only the first statement, as the proof for both statements is essentially the same.

Once we have established the claimed equivalence of automorphism groups, we get

$$\operatorname{Aut}(\underline{V}) \simeq \Omega\,\mathbf{B}\operatorname{Aut}(\underline{V}) \simeq \underline{\Omega\,\mathbf{B}\operatorname{Aut}(V)} \simeq \underline{\operatorname{Aut}(V)}$$

for every $V \in \mathrm{Ani}$, which directly implies the asserted commutativity of the square above.

By assumption there are classifiers $\widehat{\mathrm{Obj}}_\kappa^{\mathrm{Ani}} \to \mathrm{Obj}_\kappa^{\mathrm{Ani}}$ and $\widehat{\mathrm{Obj}}_\kappa^{\mathcal{X}} \to \mathrm{Obj}_\kappa^{\mathcal{X}}$ for relatively κ-compact morphisms in Ani and $\mathcal{X}$, respectively. Let $V \in \mathrm{Ani}$ be a κ-compact object. Thus, $V \simeq \widehat{\mathrm{Obj}}_\kappa^{\mathrm{Ani}} \times_{\mathrm{Obj}_\kappa^{\mathrm{Ani}}} *$ by Ex. 4.3.3. Furthermore, by Cor. 4.4.3, the following diagram in $\mathcal{X}$ consists of pullback squares

$$\begin{tikzcd}
\underline{V} \arrow[r] \arrow[d] & \underline{\widehat{\mathrm{Obj}}_\kappa}^{\mathrm{Ani}} \arrow[r] \arrow[d] & \widehat{\mathrm{Obj}}_\kappa^{\mathcal{X}} \arrow[d] \\
\underline{*} \simeq 1_{\mathcal{X}} \arrow[r] & \underline{\mathrm{Obj}_\kappa}^{\mathrm{Ani}} \arrow[r] & \mathrm{Obj}_\kappa^{\mathcal{X}}
\end{tikzcd}$$

Since $\underline{\Delta}$ preserves effective epi- and monomorphisms, the epi/mono factorization of arrows in the lower row gives rise to a commutative diagram

$$\begin{tikzcd}
1_{\mathcal{X}} \arrow[r, two heads] \arrow[dr, two heads] & \underline{\mathbf{B}\,\mathrm{Aut}(V)} \arrow[r, hook] & \underline{\mathrm{Obj}_\kappa}^{\mathrm{Ani}} \arrow[d] \\
 & \mathbf{B}\,\mathrm{Aut}(\underline{V}) \arrow[r, hook] & \mathrm{Obj}_\kappa^{\mathcal{X}}
\end{tikzcd}$$

Applying Prop. 2.3.16 to the composition $\underline{\mathbf{B}\,\mathrm{Aut}(V)} \to \mathrm{Obj}_\kappa^{\mathcal{X}}$, yields a epi/mono factorization

$$\underline{\mathbf{B}\,\mathrm{Aut}(V)} \twoheadrightarrow X \hookrightarrow \mathrm{Obj}_\kappa^{\mathcal{X}}$$

Precomposing this morphism with $1_{\mathcal{X}} \twoheadrightarrow \underline{\mathbf{B}\,\mathrm{Aut}(V)}$ produces a different epi/mono factorization of the morphism $1_{\mathcal{X}} \to \mathrm{Obj}_\kappa^{\mathcal{X}}$. Thus, X has to be equivalent to $\mathbf{B}\,\mathrm{Aut}(\underline{V})$ by uniqueness of epi/mono factorizations. This results in the following diagram

$$\begin{tikzcd}
1_{\mathcal{X}} \arrow[r, equal] \arrow[d, two heads] & 1_{\mathcal{X}} \arrow[d, two heads] \\
\underline{\mathbf{B}\,\mathrm{Aut}(V)} \arrow[r, two heads] & \mathbf{B}\,\mathrm{Aut}(\underline{V})
\end{tikzcd}$$

Using Lem. 2.3.17, one concludes, that the lower horizontal arrow is an equivalence. ☐

Corollary 4.4.6. *Let $V \in \mathcal{X}$ be κ-compact for a sufficiently large regular cardinal κ and suppose V lies in one of the following full subcategories*

- $\mathcal{X}^{\mathrm{lc}}$ *i.e. V is locally constant;*
- $\mathcal{X}^{\mathrm{flc}}$ *i.e. V is finitely locally constant;*
- $\mathcal{X}^{\mathrm{lissé}}$ *i.e. V is finitely locally constant constructible (i.e. lisse).*

Then, $\mathbf{B}\,\mathrm{Aut}(V)$ *and* $\mathrm{Aut}(V)$ *lie in this full subcategory, as well.*

Proof. If V is locally constant, then there are objects $U_i \in \mathcal{X}$ for some indexing set I, such that the canonical map $\coprod_{i\in I} U_i \twoheadrightarrow 1_{\mathcal{X}}$ is an effective epimorphism and $V \times U_i \simeq \underline{K_i} \times U_i$ for some $K_i \in \mathrm{Ani}$. Thus, Prop. 4.4.5 implies

$$\mathbf{B}\,\mathrm{Aut}(V) \times U_i \simeq \mathbf{B}\,\mathrm{Aut}(V \times U_i) \simeq \mathbf{B}\,\mathrm{Aut}(\underline{K_i} \times U_i) \simeq \mathbf{B}\,\mathrm{Aut}(\underline{K_i}) \times U_i \simeq \underline{\mathbf{B}\,\mathrm{Aut}(K_i)} \times U_i.$$

Hence, $\mathbf{B}\operatorname{Aut}(V)$ and also $\operatorname{Aut}(V)$ are locally constant.

If each K_i is π-finite, then each $\mathbf{B}\operatorname{Aut}(K_i) \in \mathrm{Ani}$ is π-finite, since this is the classifying space of the usual automorphism group as found in [May92] (see Ex. 4.3.7). This proves the statement for $V \in \mathcal{X}^{\mathrm{flc}}$. Now, the case $V \in \mathcal{X}^{\mathrm{lissé}}$ is obvious, since we still use the same **finite** indexing set I in the argument above to exhibit $\mathbf{B}\operatorname{Aut}(V)$ as element of $\mathcal{X}^{\mathrm{flc}}$. □

4.5 Gerbes and Bands

The previous discussion of Eilenberg-MacLane objects was subsumed by the assumption, that these objects are pointed. Dropping this assumption the matter gets more complicated, although not totally unreachable. Again, we follow [NSS14] for the outline of general, non-abelian gerbes, continuing the section for ∞-bundles, but later specialise to the more restrictive treatment in [HTT, §7.2.2] for the case of abelian $\mathcal{EM}$ gerbes.

Definition 4.5.1 ([NSS14, 4.41])**.** Let $1 \leq n$. An object E of an ∞-topos $\mathcal{X}$ is an **n-gerbe** if E is connected and n-truncated. This definition is extended to $n = \infty$ simply by referring to all connected objects of $\mathcal{X}$ as ∞-gerbes. An Eilenberg-MacLane (short $\mathcal{EM}$) n**-gerbe** in $\mathcal{X}$ is an n-gerbe which is $n-1$-connected.

The next Lemma constructs a natural example of $\mathcal{EM}$-gerbes, as fiber of morphisms appearing in a Postnikov tower.

Lemma 4.5.2. *Let $X \in \mathcal{X}$ be an object, $k \geq -1$ and let $p\colon 1_{\mathcal{X}} \to \tau_{\leq k}X$ be a map in $\mathcal{X}$. Then, the fiber V of $\tau_{\leq k+1}X \to \tau_{\leq k}X$ over p is an $\mathcal{EM}$-(k+1)-gerbe, whose only non-trivial homotopy group is equivalent to the image of $\pi_{k+1}X$ under $\mathcal{X}_{/X} \to \mathcal{X}_{/V}$.*

Proof. By Definition, V is the pullback

$$\begin{tikzcd}
V \arrow[r, "f"] \arrow[d, "t"] & \tau_{\leq k+1}X \arrow[d, "g"] \\
1_{\mathcal{X}} \arrow[r, "p"] & \tau_{\leq k}X
\end{tikzcd}$$

where g is k-connected by Cor. 2.8.6. Hence, $t\colon V \to 1_{\mathcal{X}}$ is k-connected as connected morphisms are stable under pullback (Lem. 2.8.7). Consider the essential geometric morphism $g^*\colon \mathcal{X}_{/\tau_{\leq k}X} \to \mathcal{X}_{/\tau_{\leq k+1}X}$. Since $p\colon 1_{\mathcal{X}} \to \tau_{\leq k}X$ is a $k+1$-truncated object in $\mathcal{X}_{/\tau_k X}$, Lem. 2.6.4 exhibits $g^*(p) \simeq (V \to \tau_{\leq k+1}X)$ as a $k+1$-truncated object of $\mathcal{X}_{/\tau_{\leq k+1}X}$. Hence, $\pi_n V \simeq f^*\pi_n(\tau_{\leq k+1}X)$ in $\mathcal{X}_{/V}$ where $f^*\colon \mathcal{X}_{/\tau_{k+1}X} \to \mathcal{X}_{/V}$ is the morphism induced by pulling back along f. □

Note. The statement above can also be proved using the long exact sequence of [HTT, 6.5.1.5] associated to the fiber sequence $V \to \tau_{\leq k+1}X \to \tau_{\leq k}X$.

Similar to categories and topoi, it suffices to treat the case of ∞-gerbes, as the case of n-gerbes is recovered directly from the discussion of ∞-gerbes by truncation.

Definition 4.5.3 ([NSS14, Def.4.48])**.** Let $\mathcal{X}$ be an ∞-topos. Given a group object $G \in \mathrm{Grp}_{\mathbb{E}_1}(\mathcal{X})$, we refer to the $\mathbf{B}G$-fiber ∞-bundles over $1_{\mathcal{X}}$ as **G-∞-gerbes** and denote $G\mathrm{Gerbe}_{\mathcal{X}} := \mathbf{B}G\text{-}\mathrm{Bun}(1_{\mathcal{X}})$. Unravelling Def. 4.3.1, any G-∞-gerbe E is characterized by

1) an effective epimorphism $U \to 1_{\mathcal{X}}$

2) and an equivalence $E \times U \simeq (\mathbf{B}G) \times U$.

An G-∞-gerbe is said to be **abelian**, if $G \in \mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{X})$.

Warning 4.5.4. There is deviation between the notion of a G-gerbe in this section and in [HTT, §7.2.2]. Lurie only considers abelian $\mathcal{EM}$-gerbes. Let $A \in \mathcal{A}\mathrm{b}(\mathcal{X})$ be a discrete abelian group object and $n \geq 1$. A "n-gerbe banded by A" in [HTT] refers to a $\mathbf{B}^{n-1}A$-gerbe in the sense of Def. 4.5.3.

Via their definition as $\mathbf{B}G$-fiber bundles, the Classification of fiber bundles (Thm. 4.3.6) immediately specialises to a classification of G-gerbes by setting $V = \mathbf{B}G$.

Corollary 4.5.5 ([NSS14, Cor.4.51])**.** *Let $\mathcal{X}$ be an ∞-topos and $G \in \mathrm{Grp}_{\mathbb{E}_1}(\mathcal{X})$. Then, G-∞-gerbes are classified by $\mathbf{Aut}(\mathbf{B}G)$-cohomology:*

$$\pi_0 G\mathrm{Gerbe} \simeq H^1(X, \mathbf{Aut}(\mathbf{B}G))\,.$$

Note. This directly extends the classification of 1-gerbes of [Gir71] (cf. [NSS14, 4.52]).

In [Gir71], Giraud associated a so called *band* (fr. *lien*) to each gerbe and defined the second cohomology class $H^2_{\mathrm{Gir}}(L)$ for a band L as the set of cartesian equivalence classes of gerbes with band L. Those gerbes with band L, which admit a global section, all fall in one equivalence class, which forms the neutral element of $H^2_{\mathrm{Gir}}(L)$. Although H^2_{Gir} is not functorial in general, it is case of taking fibres. This phenomenon was used by Chough [Cho22, 4.12] to establish an explicit special case of Thm. 4.6.2 serving as key step toward the non-abelian proper base change theorem [Cho22, 1.2]. We will take a different route. Indeed, [NSS14] provides a different notion of *bands*, which is more restrictive, yet completely sufficient for our cause. The upshot to the restriction is its functoriality and the fact that bands of G-gerbes with discrete G live in the first cohomology group of a principal bundle with **discrete** action group (see Def. 4.5.7). The proof of Thm. 4.6.2 will rely on this fact rather than the theory in [Gir71].

Definition 4.5.6. Let $G \in \tau_{\leq k}\,\mathrm{Grp}_{\mathbb{E}_1}(\mathcal{X})$ be a k-group object of $\mathcal{X}$ for some $k \geq 0$. Define the **outer automorphism group** of G to be

$$\mathrm{Out}(G) := \tau_{\leq k}(\mathrm{Aut}(\mathbf{B}G))$$

Definition 4.5.7 (Bands)**.** For $G \in \tau_{\leq k}\,\mathrm{Grp}_{\mathbb{E}_1}(\mathcal{X})$, the truncation $\mathbf{c}\colon \mathbf{B}\,\mathrm{Aut}(\mathbf{B}G) \to \mathbf{B}\,\mathrm{Out}(G)$ induces a morphism on cohomology of $\mathcal{X}$. Thus, the Classification 4.5.5 yields a morphism

$$\mathrm{Band}_{\mathcal{X}}\colon \pi_0 G\mathrm{Gerbe}_{\mathcal{X}} \to H^1(\mathcal{X}, \mathrm{Out}(G))$$

The elements in the essential image of this morphism are called **bands**. Thus, to every $E \in G\mathrm{Gerbe}$ we may associate a **band** $[\phi_E] \in H^1(\mathcal{X}, \mathrm{Out}(G)) = \pi_0\Gamma_{\mathcal{X}}(\mathbf{B}\,\mathrm{Out}(G))$.

Observation 4.5.8. An $E \in G\mathrm{Gerbe}$ admits a global section if and only if $\mathrm{Band}(E) \cong 1$ in $H^1(\mathcal{X}, \mathrm{Out}(G))$.

In the abelian case of $\mathcal{EM}$-gerbes of degree $n \geq 2$, Lurie provides a finer classification in [HTT, §7.2.2]. For any such gerbe $E \in \mathcal{X}$, the only non-trivial homotopy group $\pi_n(E) \in \mathcal{X}_{/E}$ is equivalent to an (discrete) abelian object $A \in \mathcal{A}\mathrm{b}(\mathcal{X})$ under the pullback map $\mathcal{X} \to \mathcal{X}_{/E}$ (cf. Cor. 2.8.9). By considering such equivalence $\phi\colon A \times X \to \pi_n(E)$ in $\mathcal{X}_{/E}$ as part of the data of the gerbe E and by allowing only pullback diagrams as morphisms between gerbes over arbitrary bases, which are compatible with the equivalence ϕ, Lurie is able to classify gerbes over arbitrary bases in [HTT, 7.2.2.26]. Restricting Lurie's result to gerbes over $1_{\mathcal{X}}$ allows for the following classification, which is an essential ingredients to Thm. 4.6.2.

Proposition 4.5.9 ([HTT, 7.2.2.27])**.** *Let $A \in \mathcal{A}\mathrm{b}(\mathcal{X})$ be a (discrete) abelian group object and $n \geq 2$. Then, there is a canonical bijection*

$$\theta\colon \pi_0\,\mathbf{B}^{n-1}A\text{-Gerbe}_{\mathcal{X}} \xrightarrow{\cong} H^{n+1}(\mathcal{X}, A).$$

Corollary 4.5.10 ([HTT, 7.2.2.28])**.** *Let $A \in \mathcal{A}\mathrm{b}(\mathcal{X})$. A $\mathbf{B}^{n-1}A$-gerbe E admits a global section if and only if $\theta(E) = 0 \in H^{n+1}(\mathcal{X}, A)$.*

4.6 Generalizing classical to ∞-Cohomology

In this section, a general method is developed for comparing the cohomology of ∞-topoi by extending existing identifications of classical cohomology groups. The Base Change for schemes, originally proved in [AGVb, XII.5], or the Artin comparison of algebraic and complex analytic geometry, as outlined in [AGVb, XVI.4.1], are typical examples of such classical identifications. Given the variety of similar results, the method admits numerous applications. It will serve, in particular, to derive general base change results from abelian ones in § 8.4 as well as a comparison between étale sheaves and their analytifications in § 9.3.

The following elementary Lemma promotes an isomorphism of cohomology groups, in the sense of Def. 4.1.8, to an equivalence of anima.

Lemma 4.6.1 ([Car16, 4.10])**.** *For a geometric morphism $f^* : \mathcal{F} \to \mathcal{E}$ of ∞-topoi and an $A \in \mathcal{A}\mathrm{b}(\mathcal{F})$, the following two statements are equivalent:*

1) $H^n(\mathcal{F}, A) \to H^n(\mathcal{E}, f^*A)$ is an isomorphism for all $n \geq 0$.

2) $\Gamma_{\mathcal{F}}(\mathbf{B}^n A) \xrightarrow{f^*} \Gamma_{\mathcal{E}}(\mathbf{B}^n f^*A)$ is an equivalence for all $n \geq 0$.

Proof. The implication 2) $\Rightarrow$ 1) follows directly from Def. 4.1.8.
Suppose that $H^n(\mathcal{F}, A) \cong H^n(\mathcal{E}, f^*A)$ for all $n \geq 0$.
Then, the case $n = 0$ is trivial, because A is 0-truncated.
Suppose $n \geq 1$. Since $\mathbf{B}^n A$ is connected, both anima $\Gamma_{\mathcal{F}}(\mathbf{B}^n A)$ and $\Gamma_{\mathcal{E}}(\mathbf{B}^n f^*A)$ are connected. Then, there are isomorphisms for $0 \leq k < n$:

$$\begin{aligned}\pi_k(\Gamma_{\mathcal{F}}(\mathbf{B}^n A)) &\cong \pi_0\Omega^k(\Gamma_{\mathcal{F}}(\mathbf{B}^n A))\\ &\underset{\substack{\Gamma\text{ preserves}\\ \text{limits}}}{\cong} \pi_0\Gamma_{\mathcal{F}}(\Omega^k\mathbf{B}^n A)\\ &\cong \pi_0\Gamma_{\mathcal{F}}(\mathbf{B}^{n-k}A) = H^{n-k}(\mathcal{F}, A)\end{aligned}$$

By assumption, this implies

$$\pi_k(\Gamma_{\mathcal{F}}(\mathbf{B}^n A)) \cong \pi_k(\Gamma_{\mathcal{E}}(\mathbf{B}^n f^* A)) \ \forall 0 \leq k < n$$

Since $\mathbf{B}^n A$ and $\mathbf{B}^n f^* A$ are n-truncated, one concludes for all $k \geq 0$

$$\pi_k(\Gamma_{\mathcal{F}}(\mathbf{B}^n A)) \cong \pi_k(\Gamma_{\mathcal{E}}(K(f^* A, n))) \qquad \square$$

The upcoming main Theorem of this section offers a possibility to extend a comparison of classical cohomology groups to a comparison of global sections, thus of general cohomology of ∞-topoi. I discovered Theorem. 4.6.2 independently while working through [Car16, 4.11] and [Cho22, 4.10], which both contain special cases of this general method.

Theorem 4.6.2. *Let $f^*\colon \mathcal{X} \to \mathcal{Y}$ a geometric morphism of $\mathcal{LT}\mathrm{op}$. Let $\mathcal{X}^\bullet \subseteq \tau_{<\infty}\mathcal{X}$ be a full subcategory of truncated objects in $\mathcal{X}$ satisfying the conditions:*

- *a) $\mathcal{X}^\bullet$ is stable under truncation*
- *b) For each $X \in \mathcal{X}^\bullet$ and $n \geq 1$, the image of the homotopy group $\pi_n(X) \in \tau_{\leq 0}\mathcal{X}_{/X}$ under the forgetful map $\tau_{\leq 0}\mathcal{X}_{/X} \to \tau_{\leq 0}\mathcal{X}$ lies in $\mathcal{X}^\bullet$.*
- *c) For any $G \in \mathcal{G}\mathrm{rp}(\mathcal{X}^\bullet)$, the outer automorphism group $\mathrm{Out}(G)$ is in $\mathcal{G}\mathrm{rp}(\mathcal{X}^\bullet)$.*

If f^* induces

- *1) An isomorphism $H^0(\mathcal{X}, F) \cong H^0(\mathcal{Y}, f^*F)$ for any $F \in \mathrm{Disc}(\mathcal{X}^\bullet)$;*
- *2) An equivalence of categories of principle bundles $G\mathrm{Bun}_{\mathcal{X}} \simeq f^*G\mathrm{Bun}_{\mathcal{Y}}$ for any $G \in \mathcal{G}\mathrm{rp}(\mathcal{X}^\bullet)$);*
- *3) An isomorphism $H^q(\mathcal{X}, A) \cong H^q(\mathcal{Y}, f^*A)$ for all $q \geq 2$ and for any $A \in \mathcal{A}\mathrm{b}(\mathcal{X}^\bullet)$.*

Then, for any $F \in \mathcal{X}^\bullet$ there is an equivalence of global sections

$$\Gamma_{\mathcal{X}}(F) \simeq \Gamma_{\mathcal{Y}}(f^*F).$$

Thus, $f|^*_{\mathcal{X}^\bullet}$ is an equivalence on ∞-cohomology.

The assumptions *a)-c)* on the subcategory $\mathcal{X}^\bullet$ are by no means artificial.

Example 4.6.3. For any ∞-topos $\mathcal{X}$ the full category of constant, truncated objects $\underline{\mathcal{X}}^{<\infty} \subset \tau_{<\infty}\mathcal{X}$ satisfies the conditions *a)-c)* by virtue of Cor. 2.8.10 and Prop. 4.4.5. Likewise, the categories $\mathcal{X}^{\mathrm{flc}}$ of finitely locally constant objects and $\mathcal{X}^{\mathrm{liss\acute{e}}}$ of lissé objects both satisfy the conditions *a)-c)*, because of Cor. 2.8.10 and Cor. 4.4.6.

This yields the following Corollary.

Corollary 4.6.4. *Let $f^*\colon \mathcal{X} \to \mathcal{Y}$ be a geometric morphism and let $\underline{\mathcal{X}}^\pi \subset \tau_{<\infty}\mathcal{X}$ be the full subcategory of constant objects associated to π-finite anima.*
If f^* satisfies the conditions 1) - 3) of Thm. 4.6.2 when setting $\mathcal{X}^\bullet := \underline{\mathcal{X}}^\pi$, then, f^* induces an

equivalence of profinite shapes

$$\widehat{\Pi}^{\mathcal{Y}}_{\infty} \circ f^* \overset{\simeq}{\Rightarrow} \widehat{\Pi}^{\mathcal{X}}_{\infty}$$

obtained by postcomposing the natural transformation $\epsilon_f \colon \Pi^{\mathcal{Y}}_{\infty} \circ f^* \Rightarrow \Pi^{\mathcal{X}}_{\infty}$ *of Cor. 3.2.6 with the restriction* $i^*_{\pi} \colon \mathrm{Pro}(\mathrm{An}) \to \mathrm{Pro}(\mathrm{An}^{\pi})$*.*

Question 4.6.5. In light of Corollary 2.9.11, it is natural to ask whether the subcategory $\mathcal{X}^{\mathrm{tc}} \subseteq \mathcal{X}$ of truncated coherent objects satisfies condition (c) of Theorem 4.6.2. While $\mathcal{X}^{\mathrm{tc}}$ is often stable under certain homotopical constructions, the analogue of Corollary 4.4.5 for coherent (rather than constant) objects does not appear to follow from standard methods.

Thm. 4.6.2 will be proven by combining ideas of [Cho22, Lem.4.10] and the alternative proof of Thm. 4.6.7, which is a simplified version of [Car16, Prop.4.11].

Proof of Thm. 4.6.2. Pick an arbitrary $F \in \mathcal{X}^{\bullet}$. By assumption F is n-truncated for some $n \geq -2$. Thus, we may invoke an induction on n to prove this statement.

Case n < 0:

The cases $n = -2$ and $n = -1$ are immediate, since then $\Gamma_{\mathcal{X}}(F) \simeq * \simeq \Gamma_{\mathcal{Y}}(f^*F)$.

Case n = 0:

This is exactly assumption *1)*, since $F \in \tau_{\leq 0}\mathcal{X}^{\bullet} \simeq \mathrm{N}(\mathrm{Disc}(\mathcal{X}^{\bullet}))$ and therefore $\Gamma_{\mathcal{X}}(F)$ and $\Gamma_{\mathcal{Y}}(f^*F)$ are 0-truncated.

Induction step:

First, suppose $\Gamma_{\mathcal{X}}(\tau_{\leq n-1}F) = \emptyset$. Then, by the induction hypothesis, $\Gamma_{\mathcal{Y}}(f^*\tau_{\leq n-1}F)) = \emptyset$ as well. Truncation induces the following maps of anima

$$\Gamma_{\mathcal{X}}(F) \to \Gamma_{\mathcal{X}}(\tau_{\leq n-1}F) = \emptyset \qquad \Gamma_{\mathcal{Y}}(f^*F) \to \Gamma_{\mathcal{Y}}(\tau_{\leq n-1}f^*F) \overset{2.6.4}{\simeq} \Gamma_{\mathcal{Y}}(f^*\tau_{\leq n-1}F) = \emptyset.$$

Therefore, $\Gamma_{\mathcal{X}}(F) = \emptyset = \Gamma_{\mathcal{Y}}(f^*F)$, as $\emptyset$ is the initial object in Ani.

Now, suppose $\Gamma_{\mathcal{X}}(\tau_{n-1}F) \neq \emptyset$ and fix an arbitrary point $1_{\mathcal{X}} \to \tau_{\leq n-1}F$. Let V be the fiber of $F \to \tau_{\leq n-1}F$ over this point. Applying Γ yields a diagram in Ani, commuting up to homotopy

$$\begin{tikzcd}
\Gamma_{\mathcal{X}}(V) \arrow[r] \arrow[d, "f^*"] & \Gamma_{\mathcal{X}}(F) \arrow[r] \arrow[d, "f^*"] & \Gamma_{\mathcal{X}}(\tau_{n-1}F) \arrow[d, "f^*"] \\
\Gamma_{\mathcal{Y}}(f^*V) \arrow[r] & \Gamma_{\mathcal{Y}}(f^*F) \arrow[r] & \Gamma_{\mathcal{Y}}(\tau_{n-1}f^*F).
\end{tikzcd}$$

Here, both the upper and lower row are fiber sequences in Ani, since Γ and f^* preserve finite limits. As the induction hypothesis applies to $\tau_{n-1}F$, it suffices to prove that f^* induces an equivalence $\Gamma_{\mathcal{X}}(V) \simeq \Gamma_{\mathcal{Y}}(f^*V)$. In that case, the five lemma, applied to the long exact sequences associated to the fiber sequences above, shows that f^* induces an equivalence $\Gamma_{\mathcal{X}}(F) \simeq \Gamma_{\mathcal{Y}}(f^*F)$. As V is $n-1$-connected and n-truncated, it highly depends on $\pi_n V$. As $\pi_n V$ is abelian if $n \geq 2$, we continue with two distinct cases.

Case n = 1:

Let us assume, that $\Gamma_{\mathcal{X}}(V) \neq \emptyset$ for a start. Then, V is pointed, hence an Eilenberg-MacLane object of degree 1 in $\mathcal{X}$. Let $t \colon V \to 1_{\mathcal{X}}$ be the unique map to a terminal object. By assumption *b)* the discrete group object $G \in \mathcal{X}$ corresponding to $\pi_1 V$ via $t^* \colon \mathcal{X} \to \mathcal{X}_{/X}$ lies in $\mathcal{G}\mathrm{rp}(\mathcal{X}^\bullet)$, since $G \simeq t_! \pi_1 V$ (cf. Prop. 2.8.8). Therefore, Thm. 4.2.8 together with assumption *2.* imply

$$\Gamma_{\mathcal{X}}(V) \simeq G\mathrm{Bun}_{\mathcal{X}} \simeq f^* G\mathrm{Bun}_{\mathcal{Y}} \simeq \Gamma_{\mathcal{Y}}(\mathbf{B} f^* G) \simeq \Gamma_{\mathcal{Y}}(f^* V)\,.$$

In case $\Gamma_{\mathcal{X}}(V) = \emptyset$, we have to argue, that $\Gamma_{\mathcal{Y}}(f^* V)$ is empty, too. Now, V is solely a Eilenberg-MacLane G-gerbe of degree 1. By naturality of the band construction there is essentially commutative diagram

$$\begin{tikzcd}
\pi_0 G\mathrm{Gerbe}_{\mathcal{X}} \arrow[r, "\mathrm{Band}"] \arrow[d, "f^*"] & H^1(\mathcal{X}, \mathrm{Out}(G)) \arrow[d, "\cong"] \\
\pi_0 G\mathrm{Gerbe}_{\mathcal{Y}} \arrow[r, "\mathrm{Band}"] & H^1(\mathcal{Y}, \mathrm{Out}(f^* G))
\end{tikzcd}$$

By Def. 4.5.6, $\mathrm{Out}(G)$ is an discrete group object of $\mathcal{X}^\bullet$ i.e. lies in $\mathcal{G}\mathrm{rp}(\mathcal{X}^\bullet)$ by assumption *c)*. Thus, the right vertical map in this square is an isomorphism of ordinary groups by assumption *2.* together with Cor. 4.2.9. As V is assumed to have no global section, Lem. 4.5.8 implies that $\mathrm{Band}(V) \ncong 0$ in $H^1(\mathcal{X}, \mathrm{Out}(G))$. Under consideration of the diagram above and the isomorphism therein, the band $\mathrm{Band}(f^* V)$ cannot be 0 either. Thus, $\Gamma_{\mathcal{Y}}(f^* V) = \emptyset$.

Case $\mathbf{n} \geq \mathbf{2}$:
Again, let us assume that $\Gamma_{\mathcal{X}}(V) \neq \emptyset$ i.e. that $V \simeq \mathbf{B}^n A$ for $A \in \mathcal{A}\mathrm{b}(\mathcal{X})$ corresponding to $\pi_n(V)$. Since we even have $A \in \mathcal{A}\mathrm{b}(\mathcal{X}^\bullet)$ by assumption *b)*, we conclude that $\Gamma_{\mathcal{X}}(V) \simeq \Gamma_{\mathcal{Y}}(f^* V)$ from Thm. 4.6.1 using assumption *3.*

On the other hand, if $\Gamma_{\mathcal{X}}(V) = \emptyset$, then V is a general Eilenberg-MacLane $\mathbf{B}^{n-1}A$-gerbe in $\mathcal{X}$ of degree $n \geq 2$. Using Cor. 4.5.10, we conclude that $H^{n+1}(\mathcal{X}, A) \ncong 0$, as V does not admit a global section. Moreover, we have $H^{n+1}(\mathcal{X}, A) \cong H^{n+1}(\mathcal{Y}, f^* A)$ by assumption *3.* This implies, $H^{n+1}(\mathcal{Y}, f^* A) \ncong 0$ and in conclusion $f^* V$ does not admit a global section either. ☐

Remark 4.6.6. It was brought to me that Haine, Holzschuh and Wolf have established a Theorem ([HHW24, 2.15], which is somewhat similar to Thm. 4.6.2. However, I had no knowledge of the paper [HHW24] at the time of writing this section. Furthermore, there are several differences between Thm. 4.6.2 and [HHW24, Proposition 2.15]. First, the results in [HHW24] require more restrictive assumptions on the subcategory $\mathcal{X}^\bullet$ – such as stability under finite limits – to satisfy the axioms of an “étale coefficient subcategory” [HHW24, Def. 2.20]. More significantly, whereas the approach in [HHW24] relies on the classical theory of non-abelian cohomology [Gir71] by assuming the comparison holds for all 1-truncated objects, Theorem 4.6.2 utilizes the modern framework of ∞-bundles and gerbes (cf. § 4.5). This allows for a more flexible formulation that does not require a prior comparison of all 1-truncated objects.

To establish Corollary 4.6.4 in certain geometric settings, the full machinery of ∞-gerbes can be bypassed in favor of a more direct argument. Specifically, when restricting to sheaves over locally connected sites, the comparison of profinite shapes from Corollary 4.6.4 admits a simpler proof under slightly distinct assumptions, as formulated below.

Theorem 4.6.7 (Profinite Comparison)**.** *Let $(\mathcal{C}, \sigma)$ and $(\mathcal{D}, \tau)$ be ∞-presites, such that $\mathrm{Shv}_\sigma(\mathcal{C})$ and $\mathrm{Shv}_\tau(\mathcal{D})$ are locally connected. Let $f \colon (\mathcal{C}, \sigma) \to (\mathcal{D}, \tau)$ be a morphism of presites, whose* *associated geometric morphism* f^* *induces*

1) *An isomorphism* $\Pi_0^{\mathcal{Y}} \circ f^* \xRightarrow{\cong} \Pi_0^{\mathcal{X}}$ *in* $\mathrm{Fun}(\mathrm{Disc}(\mathcal{X}), \mathrm{Set})$.

2) *An equivalence* $\underline{G}\mathrm{Bun}_{\mathrm{Shv}_\sigma(\mathcal{C})} \xrightarrow{\simeq} \underline{G}\mathrm{Bun}_{\mathrm{Shv}_\tau(\mathcal{D})}$ *for all finite groups* G.

3) *An isomorphism* $H^n(L_\sigma y(c), \underline{A}) \cong H^n(L_\tau y f(c), \underline{A})$ *for all* $n \geq 0$ *and for any finite abelian group* A.

Then, there is an equivalence of profinite shapes

$$\widehat{\Pi}_\infty^{\mathrm{Shv}_\tau(\mathcal{D}))} \circ f^* \xRightarrow{\simeq} \widehat{\Pi}_\infty^{\mathrm{Shv}_\sigma(\mathcal{C})}.$$

obtained by composing the natural transformation $\epsilon_f \colon \Pi_\infty^{\mathrm{Shv}_\tau(\mathcal{D}))} \circ f^* \Rightarrow \Pi_\infty^{\mathrm{Shv}_\sigma(\mathcal{C})}$ *of Cor. 3.2.6 with the restriction* $i_\pi^* \colon \mathrm{Pro}(\mathrm{An}) \to \mathrm{Pro}(\mathrm{An}^\pi)$.

Starting with series of reductions analogous to those in [Car16, Prop. 4.11], we complete the proof via a streamlined argument employing the fiber sequence of the proof of Theorem 4.6.2.

Proof. Note that $f^* L_\sigma y \simeq L_\tau y f$. Since every ∞-sheaf is a colimit of representables, f^* preserves colimits and natural equivalences can be checked pointwise, it suffices to show that f induces an equivalence

$$\Pi_\infty^{\mathrm{Shv}_\sigma(\mathcal{C})}(L_\sigma y(c))(V) \simeq \Pi_\infty^{\mathrm{Shv}_\tau(\mathcal{D})}(L_\tau y f(c))(V), \qquad (\boxed{\star})$$

for all $V \in \mathrm{Ani}^\pi$ and all $c \in \mathcal{C}$. Since $\mathrm{Shv}_\sigma(\mathcal{C})$ is locally connected, it suffice reduce to considering only those $c \in \mathcal{C}$, such that $L_\sigma y(c)$ is connected. In this case, $L_\tau y f(c)$ is likewise connected by assumption *1)* and Prop. 3.4.5. Therefore, the pro-anima $\Pi_\infty^{\mathrm{Shv}_\sigma(\mathcal{C})}(Lyc)$ and $\Pi_\infty^{\mathrm{Shv}_\tau(\mathcal{D})}(L_\tau y f(c))$ each preserve coproducts by Lem. 3.4.4. In sum, it suffices to show the equivalence $\boxed{\star}$ solely for connected $V \in \mathrm{An}^\pi$.

As every $V \in \mathrm{An}^\pi$ is n-truncated for an $n \in \mathbb{N}$, we may proceed with an induction on n.

Case n=0:
By assumption $V \simeq *$. Since geometric morphism $\underline{\Delta}$ preserves finite limits it follows that

$$\Pi_\infty^{\mathrm{Shv}_\sigma(\mathcal{C})}(L_\sigma y(c))(*) \overset{3.2.2}{=} \mathrm{map}_{\mathrm{Shv}_\sigma(\mathcal{C})}(L_\sigma y(c), \underline{*}) \simeq * \simeq \Pi_\infty^{\mathrm{Shv}_\tau(\mathcal{D})}(L_\tau y f(c))(*).$$

Case n=1:
By assumption, V is a 1-truncated, π-finite (path)-connected CW-complex. Thus, any (equivalent) choice of base point exhibits V as Eilenberg-MacLane space of degree 1 corresponding to a finite group G. The assertion thus follows assumption *1)* and Thm. 4.2.8.

We continue with the induction step for $n \geq 2$.

Induction Step:
By assumption, $V \in \mathrm{An}^\pi$ is connected and n-truncated for an $n \geq 2$. Let $\tau_{\leq n-1}V$ be the $n-1$-truncation of V. The fiber of the associated fibration $V \to \tau_{\leq n-1}V$ is the Eilenberg-MacLane space $\mathbf{B}^n(\pi_n V)$. Since $\Pi_\infty^{\mathcal{X}}$ preserves finite limits for every ∞-topos $\mathcal{X}$, f^* induces a diagram of fiber sequences of anima

$$\begin{tikzcd}
\Pi_\infty^{\mathrm{Shv}_\tau(\mathcal{C})}(L_\tau y f(c))(\mathbf{B}^n(\pi_n V)) \arrow[r] \arrow[d, "\overset{2)}{\simeq}"'] & \Pi_\infty^{\mathrm{Shv}_\tau(\mathcal{C})}(L_\tau y f(c))(V) \arrow[r] \arrow[d] & \Pi_\infty^{\mathrm{Shv}_\tau(\mathcal{C})}(L_\tau y f(c))(\tau_{\leq n-1} V) \arrow[d, "\simeq"] \\
\Pi_\infty^{\mathrm{Shv}_\sigma(\mathcal{C})}(L_\sigma y(c))(\mathbf{B}^n(\pi_n V)) \arrow[r] & \Pi_\infty^{\mathrm{Shv}_\sigma(\mathcal{C})}(L_\sigma y(c))(V) \arrow[r] & \Pi_\infty^{\mathrm{Shv}_\sigma(\mathcal{C})}(L_\sigma y(c))(\tau_{\leq n-1} V).
\end{tikzcd}$$

By the induction hypothesis, the right vertical map is an equivalence and by assumption *2)*, the left map vertical map is an equivalence. By considering the corresponding morphism of long exact sequences, it follows that the middle vertical map is an equivalence as well. ☐

Corollary 4.6.8. *In the situation of Cor. 4.6.4 or Thm. 4.6.7 the given geometric morphism* $f^*\colon \mathcal{X} \to \mathcal{Y}$*, induces an equivalence of pro-anima*

$$\widehat{\Pi}_\infty \mathcal{Y} \overset{\simeq}{\Rightarrow} \widehat{\Pi}_\infty \mathcal{X}$$

since f^* *preserves terminal objects. In particular, it follows from Thm. 3.3.20, that* f^* *induces an equivalence of lissé objects*

$$\mathcal{X}^{\text{lissé}} \simeq \mathcal{Y}^{\text{lissé}}.$$

Lastly, we list a restriction of Thm. 4.6.7 to situations, where a comparison is only available for abelian cohomology groups.

Theorem 4.6.9. *Let* $(\mathcal{C}, \sigma)$ *and* $(\mathcal{D}, \tau)$ *be* ∞*-presites, such that* $\mathrm{Shv}_\sigma(\mathcal{C})$ *and* $\mathrm{Shv}_\tau(\mathcal{D})$ *are locally connected. Let* $f\colon (\mathcal{C}, \sigma) \to (\mathcal{D}, \tau)$ *be a morphism of presites, whose associated geometric morphism* f^* *induces*

1) An isomorphism $\Pi_0^{\mathcal{Y}} \circ f^* \overset{\simeq}{\Longrightarrow} \Pi_0^{\mathcal{X}}$ *in* $\mathrm{Fun}(\mathrm{Disc}(\mathcal{X}), \mathrm{Set})$.

2) An isomorphism $H^n(L_\sigma y(c), \underline{A}) \cong H^n(L_\tau y f(c), \underline{A})$ *for all* $n \geq 0$ *and for any finite abelian group* A.

Then, there is an equivalence of abelian profinite shapes (3.2.3).

$${}^{\mathrm{ab}}\widehat{\Pi}_\infty^{\mathrm{Shv}_\tau(\mathcal{D}))} \circ f^* \overset{\simeq}{\Longrightarrow} {}^{\mathrm{ab}}\widehat{\Pi}_\infty^{\mathrm{Shv}_\sigma}(\mathcal{C}).$$

obtained by composing $\epsilon_f\colon \Pi_\infty^{\mathrm{Shv}_\tau(\mathcal{D}))} \circ f^* \Rightarrow \Pi_\infty^{\mathrm{Shv}_\sigma(\mathcal{C})}$ *with* i_π^* *and* i_{ab}^*.

Proof. Proceeding as in the proof of Thm. 4.6.7, it suffices to show that ϵ_f induces an equivalence

$$\Pi_\infty^{\mathrm{Shv}_\sigma(\mathcal{C})}(L_\sigma y(c))(V) \simeq \Pi_\infty^{\mathrm{Shv}_\tau(\mathcal{D})}(L_\tau y f(c))(V)$$

for all connected $V \in \mathrm{Ani}^\pi$ with abelian fundamental group, and all $c \in \mathcal{C}$ such that $L_\sigma y(c)$ is connected. Since $\pi_1(\underline{V}) \in \tau_{\leq 0} \mathrm{Shv}_\sigma(\mathcal{C})$ is abelian, the argument of Thm. 4.6.7 applies directly, without needing to distinguish the case $n = 1$. ☐

Thm. 4.6.7 and Cor. 4.6.4 provide very explicit descriptions of the shape functor, if the ∞-topos in the codomain happens to be locally ∞-connected. The Artin Comparison ([AGVb]) provides a typical application of Thm. 4.6.2, relating the ∞-cohomology of finitely constructible sheaves on schemes locally of finite type over $\mathbb{C}$ to their counterparts in complex analytic geometry. Since analytic spaces are locally contractible, their associated ∞-topos is locally ∞-connected. In the next section, we provide a brief summary on complex analytic spaces. Since the announced results carry over to analytic spaces over non-Archimedean fields, this summary is followed by an introduction to non-Archimedean analytic geometry.

5 Complex Analytic Geometry

Definition 5.0.1. Let $\mathrm{AnAff}_{\mathbb{C}}$ denote the category of **affine analytic space**, whose objects are subsets $V \subset \mathbb{C}^n$ for some $n \in \mathbb{N}$, which are the common vanishing locus of finitely many holomorphic functions and are equipped with the subspace topology. The morphisms $f : V \to W$ in $\mathrm{AnAff}_{\mathbb{C}}$ are **holomorphic maps** i.e. if $W \subset \mathbb{C}^m$, then every point $x \in V$ admits an open neighbourhood $U \subset V$ and holomorphic functions $f_1, ..., f_m \colon U \to \mathbb{C}$ such that $f|_U = (f_1, ..., f_m)$.

Similarly, let $\mathrm{Al}_{\mathbb{C}} \subset \mathrm{AnAff}_{\mathbb{C}}$ be the category of **complex affine algebraic spaces**. The objects are subsets $X \subset \mathbb{C}^k$, which are the common vanishing locus of finitely many polynomials $f_1, ..., f_m \in \mathbb{C}[T_1, ..., T_k]$ and are equipped with the subspace topology. The morphisms $f : X \to Y$ in $\mathrm{Al}_{\mathbb{C}}$ are polynomial maps i.e. if $Y \subset \mathbb{C}^n$, then there must be polynomials $P_1, \dots, P_n \in \mathbb{C}[T_1, \dots, T_k]$ such that $f(x) = (P_1(x), \dots, P_n(x))$ for all $x \in X$.

Observation 5.0.2. $\mathrm{Al}_{\mathbb{C}}$ and $\mathrm{AnAff}_{\mathbb{C}}$ are closed under finite limits and finite coproducts. Moreover, $(\mathrm{Al}_{\mathbb{C}}, \tau_{\mathrm{open}})$ and $(\mathrm{AnAff}, \tau_{\mathrm{open}})$ both form presites.

Definition 5.0.3. Let $X \in \mathrm{Aff}_{\mathbb{C}}^{\mathrm{fty}}$ be an affine $\mathbb{C}$-scheme of finite type. Define the analytification X^{an} of X to be the (set of) $\mathbb{C}$-valued points $X(\mathbb{C})$ [36] equipped with the subset topology of $\mathbb{C}^n$. Assigning X its analytification X^{an} yields an essentially surjective functor

$$(\text{-})^{\mathrm{an}} : \mathrm{Aff}_{\mathbb{C}}^{\mathrm{fty}} \to \mathrm{Al}_{\mathbb{C}}.$$

Observation 5.0.4. If $V \subset W \in \mathrm{Al}_{\mathbb{C}}$ is the subset of a common vanishing locus of finitely many polynomials, then $V \in \mathrm{Al}_{\mathbb{C}}$ is a closed subspace of W. Since all objects in $\mathrm{Aff}_{\mathbb{C}}^{\mathrm{fty}}$ are Noetherian, every closed subscheme of an $X \in \mathrm{Aff}_{\mathbb{C}}^{\mathrm{fty}}$ corresponds to an finitely generated ideal. Consequently, $(\text{-})^{\mathrm{an}} : \mathrm{Aff}_{\mathbb{C}}^{\mathrm{fty}} \to \mathrm{Al}_{\mathbb{C}}$ sends closed immersions to closed embeddings. Therefore, $(\text{-})^{\mathrm{an}}$ sends open immersions to open embeddings.

Observation 5.0.5. $(\text{-})^{\mathrm{an}} : \mathrm{Aff}_{\mathbb{C}}^{\mathrm{fty}} \to \mathrm{Al}_{\mathbb{C}}$ sends connected affine schemes to connected topological spaces and preserves coproducts.

Lemma 5.0.6. $(\text{-})^{\mathrm{an}} : \mathrm{Aff}_{\mathbb{C}}^{\mathrm{fty}} \to \mathrm{Al}_{\mathbb{C}}$ *sends étale morphisms to local biholomorphisms.*

Proof. Consider two affine schemes $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$. An étale morphism $f : X \to Y$ corresponds to the canonical morphism

$$\phi \colon B \to \frac{B[X_1, ..., X_m]}{f_1, ..., f_n}$$

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[36] i.e for $X = \mathrm{Spec}(\frac{\mathbb{C}[X_1,...,X_n]}{(f_1,...f_r)})$ for some $n, r \in \mathbb{N}$ and polynomials $f_1, .., f_r \in \mathbb{C}[X_1, ..., X_n]$, let $X(\mathbb{C}) := c \in \mathbb{C}^n | f_1(c) = ... = f_r(c) = 0$.

for some $m, n \in \mathbb{N}$ and $f_i \in B[X_1, ..., X_n]$. Therefore, X^{an} is given by

$$X^{\mathrm{an}} = \{(y, c) \in Y^{\mathrm{an}} \times \mathbb{C}^m \mid f_1(c) = ... = f_n(c) = 0\}$$

Étale morphisms are smooth of relative dimension 0. Equivalently, the Jacobian of ϕ invertible i.e. $\det \big(\frac{\partial f_j}{\partial x_i}\big)_{i,j} \neq 0$ (cf. Lem C.4.3). Thus, for every $x \in X$ the map

$$\tilde{f} \colon X^{\mathrm{an}} \hookrightarrow Y^{\mathrm{an}} \times \mathbb{C}^m \xrightarrow{(id_{Y^{\mathrm{an}}}, f_1, ..., f_n)} Y^{\mathrm{an}} \times \{0\} = Y^{\mathrm{an}}$$

is a biholomorphism on a sufficiently small neighbourhood of x by the analytic implicit function theorem ([GR65, I.B.5, p.14]). ☐

Lemma 5.0.7. *The functor* (-)$^{\mathrm{an}}$ *preserves finite limits.*

Proof. It suffices to show that (-)$^{\mathrm{an}}$ preserves terminal objects and pullbacks as explained in [Lan78, p.112,118] or [HTT, 4.4.2.5].
The first claim is trivial, since by Definition $\mathrm{Spec}(\mathbb{C})^{\mathrm{an}} = \mathbb{C}^0 = *$.
Since 'being finite type' has the cancellation property (cf. [GW20, 10.7]) all morphisms in $\mathrm{Aff}_{\mathbb{C}}^{\mathrm{fty}}$ are of finite type. Let $A = \mathbb{C}[X_1, ..., X_k]/(\underline{f})$ be a finite type $\mathbb{C}$-algebra and

$$B = A[Y_1, ..., Y_l]/(\underline{g}) \leftarrow A \rightarrow C = A[Z_1, ..., Z_m]/(\underline{h})$$

two morphisms of finite type. Then,

$B \otimes_A C = A[\underline{Y}, \underline{Z}]/(\underline{g}, \underline{h}) = \mathbb{C}[\underline{X}, \underline{Y}, \underline{Z}]/(\underline{f}, \underline{g}, \underline{h})$.
Hence, $(\mathrm{Spec}(B) \times_{\mathrm{Spec}(A)} \mathrm{Spec}(C))^{\mathrm{an}} = \{(x, y, z) \in \mathbb{C}^{k+l+m} \mid \underline{f}(x) = 0,\ \underline{g}(x, y) = 0,\ \underline{h}(x, z) = 0\}$

On the other hand, the analytification $M := (\mathrm{Spec}(B) \times_{\mathrm{Spec}(A)} \mathrm{Spec}(C))^{\mathrm{an}}$ describes precisely the pullback in the square

$$\begin{tikzcd}
M \arrow[rr] \arrow[dd] & & \mathrm{Spec}(B)^{\mathrm{an}} \arrow[d, equal] \\
& & \{(x,y) \in \mathbb{C}^{k+l} \mid \underline{f}(x) = 0,\ \underline{g}(x,y) = 0\} \arrow[d] \\
\mathrm{Spec}(C)^{\mathrm{an}} = \{(x,z) \in \mathbb{C}^{k+m} \mid \underline{f}(x) = 0,\ \underline{h}(x,z) = 0\} \arrow[rr] & & \mathrm{Spec}(A)^{\mathrm{an}} = \{x \in \mathbb{C}^k \mid \underline{f}(x) = 0\}
\end{tikzcd}$$

☐

Corollary 5.0.8. *The analytification* (-)$^{\mathrm{an}} \colon \mathrm{Aff}_{\mathbb{C}}^{\mathrm{fty}} \rightarrow \mathrm{Al}_{\mathbb{C}}$ *induces a morphism of presites* $(\mathrm{Aff}_{\mathbb{C}}^{\mathrm{fty}}, \acute{e}t) \rightarrow (\mathrm{Al}_{\mathbb{C}}, \mathrm{lhomeo})$. *By Thm. 2.3.23, this morphism lifts to a geometric morphism in* $\mathcal{LT}\mathrm{op}$ *which we denote by* (-)$^{\mathrm{an}}$ *as well:*

$$(\text{-})^{\mathrm{an}} \colon \mathrm{Shv}_{\acute{e}t}(\mathrm{Aff}_{\mathbb{C}}^{\mathrm{fty}}) \rightarrow \mathrm{Shv}_{\mathrm{lhomeo}}(\mathrm{Al}_{\mathbb{C}}) \overset{2.6.15}{\simeq} u\,\mathrm{Shv}_{\mathrm{open}}(\mathrm{Al}_{\mathbb{C}}).$$

For completeness let us introduce the classical notions of complex analytic varieties and spaces.

Definition 5.0.9. Let $V \subset \mathbb{C}^n$ be the common vanishing locus of holomorphic functions $f_1, ..., f_n \colon \mathbb{C}^n \to \mathbb{C}$. Let $\mathcal{O}_{\mathbb{C}^n}$ denote the sheaf of germs of holomorphic functions at 0 (see [Dun18, §3.1] for instance) and $\mathcal{J} \subset \mathcal{O}_n$ the ideal sheaf generated $f_1, ..., f_n$. Equipping V with the sheaf $\mathcal{O}_V := \mathcal{O}_{\mathbb{C}}/\mathcal{J}$ produces a ringed space $(V, \mathcal{O}_V)$. From now on, affine analytic spaces always regarded to be equipped with this structure sheaf.

Classically a complex analytic space is defined as follows

Definition 5.0.10 ([GR65, V.6])**.** A general **complex analytic space** is a locally ringed space $(X, \mathcal{O}_X)$, where X is Hausdorff and every point $x \in X$ has a neighbourhood $U \subset X$, such that $(U, \mathcal{O}_X|_U)$ is isomorphic to an affine analytic space. Let $\mathrm{AnSp}_{\mathrm{C}}$ denote the category of complex analytic spaces defined as a full subcategory of the category of locally ringed spaces LocRinged.

Remark 5.0.11 (Structured Spaces, cf.[DAGV, 2.5.16, 2.5.22, 2.6.9],[Toë05, 2.2.2])**.** The definition of an analytic space (Def. 5.0.10) is fundamentally aligned with the modern treatment of schemes. The general *functor of points* – or *functorial geometry*[37] – paradigm offers an unifying approach to realize both schemes and complex analytic spaces as sheaves on a category of “affine” models $\mathcal{C}$ (such as Aff or AnAff), equipped with a Grothendieck topology τ and a distinguished class of morphisms defining a so called *geometric context* $\mathfrak{G}$, which specifies the “structure preserving morphisms”. In case of Aff, the open immersions or étale morphisms each specify a geometric context (see Ex. 5.0.12).

More precisely, given a site $(\mathcal{C}, \tau)$ and geometric context $\mathfrak{G}$, a sheaf $F \in \mathrm{Shv}_\tau(\mathcal{C})$ constitutes a *structured space*[38] (i.e. a “scheme-like” structure) if it admits a $\mathfrak{G}$-covering by representables. That is, there is a small collection of objects $\{c_i\}_{i \in I}$ in $\mathcal{C}$ and morphisms $\{Ly(c) \to F\}$ such that

- The induced map $\coprod_{i \in I} Ly_{c_i} \to F$ is an effective epimorphism in $\mathrm{Shv}_\tau(\mathcal{C})$.
- Each morphism $Ly_{c_i} \to F$ behaves like an open immersion with respect to $\mathfrak{G}$. See [Toë05, p.10] for details.

This perspective shifts the focus from the underlying underlying set of points of a scheme/analytic space to its “probes”: Every structured space is uniquely characterized by the way it is perceived by the affine test objects i.e. by the objects in $\mathcal{C}$. A concise introduction to this viewpoint is given in [Toë05]. For ∞-sites (plus geometry) this results in the general setup of derived algebraic geometry as developed by [DAGV] or [TV04].

The following examples demonstrate the versatility of this framework, recovering various classical settings from specific choices of the triple $(\mathcal{C}, \tau, \mathfrak{G})$.

Example 5.0.12.

- ($\{$open subspaces of $\mathbb{C}^n \mid \mathrm{n} \in \mathbb{N}\}, \tau_{\mathrm{open}}$, $\{$holomorphic maps with surjective differential$\}$) produces complex analytic spaces as 0-localic structured spaces.
- ($\{$open subspaces of $\mathbb{R}^n \mid n \in \mathbb{N}\}$with $\mathcal{C}^\infty$-maps, τ_{open}, $\{$local diffeomorphisms$\}$) produces the category of differentiable varieties.

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[37] This is name appeared already in Grothendieck's famous talk [Gro73].

[38] Toën [Toë05] uses the term “variety” for the case $F \in \mathrm{Shv}_\tau(\mathcal{C}; \mathrm{Set})$.

- The 0-localic structured spaces wrt. $(\mathrm{Aff}_{\mathbb{C}}, \tau_{Zar}, \{\text{open immersions}\})$ are the classical schemes over $\mathbb{C}$ ([DAGV, 2.5.16]). Restricting to $\mathrm{Aff}_{\mathbb{C}}^{\mathrm{fty}}$ the ∞-category of 0-localic structured spaces is equivalent to the (nerve) of the 1-category $\mathrm{Sch}_{/\mathbb{C}}^{\mathrm{lft}}$ of schemes locally of finite type over $\mathbb{C}$.

Observation 5.0.13. By virtue of the example above, the functor $(\text{-})^{\mathrm{an}} \colon \mathrm{Shv}_{\mathrm{Zar}}(\mathrm{Aff}_{\mathbb{C}}^{\mathrm{fty}}) \to \mathrm{Shv}_{\mathrm{open}}(\mathrm{AnAff})$ restricts to a functor

$$(\text{-})^{\mathrm{an}} \colon \mathrm{Sch}_{/k}^{\mathrm{lft}} \to \mathrm{AnSp}_{\mathbb{C}}.$$

This functor preserves finite limits, open immersions and sends étale morphisms to local biholomorphisms.

Example 5.0.14. By definition, every point $p \in X^{\mathrm{an}}$, for a $X \in \mathrm{Sch}_{/\mathbb{C}}^{\mathrm{lft}}$, corresponds to a morphism $\mathrm{Spec}(\mathbb{C}) \to X$. Since $\mathbb{C}$ is algebraically closed, the geometric points of X (see Def. C.5.1) are precisely the points of X^{an}.

Definition 5.0.15 ([Che+94, p.61,p.70])**.** Similar to algebraic geometry the dimension of an $X \in \mathrm{AnSp}_{\mathrm{C}}$ at a point $x \in X$ is defined as the Krull dimension of $\mathcal{O}_{X,x}$ i.e.

$$\dim_x X := \dim \mathcal{O}_{X,x}.$$

The global dimension of X is defined as

$$\dim(X) := \sup_{x \in X} \dim_x X.$$

The codimension of analytic subsets $A \subset B \subset X$ is defined as

$$\operatorname{codim}(A, B) := \min_{x \in A}(\dim_x B - \dim_x A).$$

This definition is equivalent to all reasonable notions of dimension (cf.[Che+94, I.10.3, I.10.12]).

6 Non-Archimedean Analytic Geometry

The purpose of this section is to give a quick overview of the analytic geometry developed by Berkovich ([Ber90], [Ber93] and [Ber95]), in order to prove the classic comparison for abelian cohomology over $\mathbb{C}$ or a non-archimedean field simultaneously in the following sections. The reader familiar with Berkovich first book [Ber90] may note, that we incorporate the more general notion of analytic spaces used in the IHES paper [Ber93]. Following Temkin [Tem11] and Ducros [Duc11] we give a concise modern outline of this theory.

The reader familiar with Lurie's derived and spectral algebraic geometry should note, that Berkovich geometry does not particularly fit into this framework. However, more recently Ben-Bassat and Kremnizer [BK17] showed, that one may think of Berkovich geometry as relative algebraic geometry. As such it naturally extends to a derived geometry. Although the corresponding functor of points perspective might offer simplifications at many places, we will not elaborate on the approach of Ben-Bassat and Kremnizer [BK17] further, due to its technical nature.

6.1 Fundamentals of Berkovich Geometry

Non-Archimedean analytic geometry originates with Tate [Tat71], whose work gave rise to the theory of Rigid Geometry. However, rigid spaces suffer from being totally disconnected. Berkovich remedies this, by an rigorous outline of the idea of [BGR84, §6.1.5] to allow disks of arbitrary polyradius, which essentially gives rise to the n-dimensional affine space [BGR84, 9.3.4/1]. This ultimately results in a framework, whose spaces are locally compact and path-connected, sitting between Rigid Geometry and the more general Adic Geometry.

For background on norms, valuations, and Banach rings, the reader is referred to [Neu92] and [Cla20]. Since the terminology differs across the literature, we briefly recall the basic definitions concerning semi-norms.

- A **real semi-norm** on an abelian group A is a function $|\text{-}| : A \to \mathbb{R}_0^+$ satisfying $|a+b| \leq |a| + |b|$, $|-a| = |a|$ and $|0| = 0$ for all $a, b \in A$. As only real (semi-)norms are considered throughout this work, the term 'real' will be omitted.

- A **semi-norm** on a ring A is a semi-norm $|\text{-}|$ on the underlying abelian group of A, such that $|ab| \leq |a||b|$ for all $a, b \in A$.

- A semi-norm $|\text{-}| : A \to \mathbb{R}^+$ is **non-Archimedean**, if it satisfies the strong triangle inequality $|a+b| \leq \max(|a|, |b|)$ for all $a, b \in A$. For a semi-normed ring A, this is equivalent to the condition, that $|n1_A| \leq 1$ (cf.[Neu92, II.3.6]).

- A morphism between semi-normed abelian groups $\phi\colon (A, |\text{-}|) \to (B, \|\text{-}\|)$ is a group morphism which is **bounded** i.e. there is a constant $C > 0$, such that $\|\phi(a)\| \leq C|a|$ for all $a \in A$. Any semi-normed group A is naturally equipped with a topology induced by the semi-metric $d(a, b) = |a - b|_A$ and any bounded homomorphism $A \to B$ of semi-normed groups is continuous (see [Cla20]) for details).

- Given a semi-normed group $(A,\ |\text{-}|)$ and a normal subgroup $H \leq A$, then the quotient A/H inherits the so called **residue semi-norm** given by

  $$\|[a]\|_H := \inf_{h \in H} |a + h| \qquad \forall\, [a] \in A/H.$$

  Thus, for every homomorphism of semi-normed groups $\varphi\colon A \to B$ the image $\varphi(A) \cong {}^{A}\!\big/_{\ker(\varphi)}$ inherits the residue semi-norm.

- A homomorphism of semi-normed groups $\varphi\colon A \to B$ is **admissible**[39], if the semi-norm on $\varphi(A)$, inherited from B, is equivalent to the residue semi-norm of φ. Observe that every admissible morphism is automatically bounded.

- A semi-norm $|\text{-}|$ on A is **power-multiplicative**, if $|a^n| = |a|^n$ for all $a \in A$. For every equivalence of power-multiplicative semi-norms $|\text{-}| \leq C\|\text{-}\|$ one can set $C = 1$, since $|a^n| \leq \sqrt[n]{C}\|a^n\|$ for all $n \geq 1$, i.e. all equivalent power-multiplicative semi-norms are equal.

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[39] Bosch, Güntzer, and Remmert [BGR84] use '**strict**' instead of 'admissible'. We follow [Tem11].

- A **(semi-)valuation** on a ring A is a multiplicative (semi-)norm on A, i.e. a semi-norm $|\text{-}|$, such that $|a \cdot b| = |a| \cdot |b|$ and $|1| = 1$.

Lemma 6.1.1. *Any non-trivial semi-valuation $|\text{-}|$ on a field K is a valuation.*

Proof. It suffices to show positive definiteness for $|\text{-}|$. Suppose there is a $x \in K \setminus \{0\}$ with $|x| = 0$, then $0 = |x||x^{-1}| = |1|$. This is a contradiction, thus $|x| \neq 0$. □

Although, all of the upcoming theory can be defined relative to a fixed arbitrary (possibly archimedean) Banach ring, we restrain ourselves for the sake of simplicity.

Convention 6.1.2. For the remainder of this section, fix a non-archimedean complete field k with norm $|\text{-}|_k : k \to \mathbb{R}_0^+$. From now on all (semi-)norms and valuations are considered non-archimedean.

Warning 6.1.3. If $(A, |\text{-}|)$ is a normed group and $H \leq A$ a normal subgroup, then $(A/H, \|\text{-}\|_H)$ is generally only a *semi*-normed group. However, if H is closed in A with respect to the topology induced by $|\text{-}|$, then the semi-norm is a *norm*.

Observation 6.1.4. Completion is bounded and functorial. Thus, it yields functors

$$\begin{aligned} \widehat{(\text{-})} &: \text{semi-normed Ab} \to \text{complete semi-normed Ab} \\ \widehat{(\text{-})} &: \text{normed } A\text{-Algebra} \to \text{Ban}_A \end{aligned}$$

where Ban_A denotes the category of Banach A-algebras for a normed ring A. This follows immediately from the universal property of the canonical map $B \to \widehat{B}$ for any semi-normed abelian group or ring B and the fact, that this map $B \to \widehat{B}$ is an isometry.

Admissible morphisms have the pleasant property that completion preserves their exactness.

Lemma 6.1.5 ([BGR84, 1.1.9/6])**.** *Let $F \xrightarrow{\phi} G \xrightarrow{\psi} H$ be an exact sequence of semi-normed groups. If ϕ and ψ are admissible, then the induced sequence $\widehat{F} \xrightarrow{\widehat{\phi}} \widehat{G} \xrightarrow{\widehat{\psi}} \widehat{H}$ is exact.*

By [Neu92, II.4] any valued field K embeds into its completion $\widehat{K}$. This embedding preserves algebraic closeness.

Lemma 6.1.6 (Krasner's Lemma, [Bos14, A.6])**.** *The completion $\widehat{K}$ of an algebraically closed non-archimedean field K is algebraically closed.*

The sum and tensor product of semi-normed modules naturally admit semi-norms, which are defined as follows.

Definition 6.1.7 ([Fre03, p.8])**.** Let A be a semi-normed ring.

- Let M be an A-module. A semi-norm $|\text{-}|_M$ on the underlying group of M is an **A-module semi norm**, if $|am|_M \leq |a|_A |m|_M$ for all $a \in A$, $m \in M$.
- For a family of semi-normed A-modules $\{(M_i, |\text{-}|_i)\}_{i \in I}$, define a semi-norm on $\oplus_{i \in I} M_i$ by setting
$$|(x_i)_{i \in I}| := \max_{i \in I} |x_i|_i$$

- If $(M, |\text{-}|_M)$ and $(N, |\text{-}|_N)$ are semi-normed A-modules, then $M \otimes_A N$ is equipped with the **tensor product semi-norm**

$$|x| := \inf_{x=\sum_{i=1}^n m_i \otimes n_i} (\max_{i=1,\dots,n} |m_i|_M |n_i|_N)$$

  where the infimum is taken over all possible representations $x = \sum_{i=1}^n m_i \otimes n_i$ with $m_i \in M$ and $n_i \in N$. The completion of this semi-normed module is denoted $M \widehat{\otimes}_A N$ and called the **completed tensor product**.

[BGR84, A.2.1.5] and [Bos14, Appendix B] verify that these notions are indeed semi-norms.

Lemma 6.1.8 ([BGR84, A.2.1.5/6])**.** *Let A be a semi-normed ring and $\{M_i\}_{i \in I}$ be family of semi-normed A-modules.*

1) $\oplus_{i \in I} M_i$ is the coproduct in the category of (semi-)normed A-modules i.e. has the usual universal property with respect to bounded maps from semi-normed A-modules.

2) If I is finite and all M_i are complete, then $\oplus_{i \in I} M_i$ is complete.

Example 6.1.9. Let $(A, |\text{-}|)$ be a non-archimedean Banach ring. Consider the polynomial ring $A[T]$. For any $r \in \mathbb{R}_{>0}$, we may define a non-archimedean norm, by setting

$$\|\sum_{i=0}^N a_i T^i\|_r := \max_i |a_i| r^i, \qquad \text{i.e. } \|T\|_r = r^{-1}.$$

Then, a formal power series $\sum_{i=0}^\infty a_i T^i$ converges with respect to $\|\text{-}\|_r$ if and only if $\lim_i |a_i| r^i = 0$. If a formal power series in $A[[T]]$ converges wrt.$\|\|_r$ one says it is convergent within radius r. In that case, one says it is strictly convergent if $r = 1$. The powerseries convergent within radius r is denoted by $A\{r^{-1}T\}$.
Then, $A\{r^{-1}T\} \subset A[[T]]$ forms a subring, since for

$$(\sum_i a_i T^i)(\sum_j b_j T^j) = \sum_{m \in \mathbb{N}^n} \sum_{i+j=m} a_i b_j T^m$$

one has

$$\lim_{m \to \infty} |\sum_{i+j=m} a_i b_j| r^m \leq \lim_{m \to \infty} \max_{i+j=m} |a_i b_j| r^m = \lim_m \max_{i+j=m} |a_i| r^i |b_j| r^j = 0$$

This example carries right over to power series in finitely many variables $A[[\underline{T}]] := A[[T_1, ..., T_n]]$. The quotients of $A\{\underline{T}^{-1}\}$ constitute the local building blocks in rigid geometry, identifying $A\{\underline{T}^{-1}\}$ with the unit ball of n dimensions. Classically, the rings $A\{\underline{r}^{-1}\underline{T}\}$ for *specific* $\underline{r} \in \mathbb{R}^n_{>0}$ are identified with the n-balls of polyradius $\underline{r}$ *after* establishing the fundamentals of rigid geometry (cf. [BGR84, 6.1.5/4]). Berkovich approach offers more flexibility by considering *all* rings $A\{\underline{r}^{-1}\underline{T}\}$ as “affines”, thus allowing n-balls of *all* polyradii right from the start.

Definition 6.1.10. Let A be a non-archimedean Banach ring. Given an $n \in \mathbb{N}$ and $\underline{r} := (r_1, ..., r_n) \in \mathbb{R}^n_{>0}$, the **free affinoid k-algebra** of dimension n with polyradius $\underline{r}$ is defined to be sub-A-algebra[40] of formal powerseries, convergent within radius $\underline{r}$

$$A\{\underline{r}^{-1}\underline{T}\} = A\{r_1^{-1}T_1, ..., r_n^{-1}T_n\} := \{\sum_{\underline{i}\in\mathbb{N}^n} a_i\, \underline{T}^i \mid \lim_{|i|\to\infty} |a_i| r_1^{i_1}\cdots r_n^{i_n} = 0\} \subset A[[T_1, ..., T_n]]\}$$

The A-algebra $A\{\underline{r}^{-1}\underline{T}\}$ is naturally endowed with the non-archimedean norm

$$||\sum_{\underline{i}} a_{\underline{i}} T^{\underline{i}}||_{\underline{r}} := \max_{\underline{i}\in\mathbb{N}^n} |a_{\underline{i}}|\, r_1^{i_1}\cdots r_n^{i_n}.$$

Example 6.1.11. Def. 6.1.10 simplifies drastically, if k carries the trivial norm. For example, if $r_1, ..., r_n \geq 1$, then $k\{\underline{r}^{-1}\underline{T}\}$ is simply $k[T_1, ..., T_n]$. On the other hand, if $r_1, ..., r_n < 1$, then $k\{\underline{r}^{-1}\underline{T}\}$ is the algebra of formal power series $k[[T_1, ..., T_n]]$ equipped with the trivial norm.

Observation 6.1.12. If the norm on A is multiplicative, then $\|\text{-}\|_r$ on $A\{r^{-1}T\}$ is a valuation for any $r \in \mathbb{R}_{>0}$.

Lemma 6.1.13. *$A\{\underline{r}^{-1}\underline{T}\}$ is a non-archimedean Banach A-algebra for any $\underline{r} \in \mathbb{R}^n_{>0}$ and non-archimedean Banach ring A.*

Proof. This is proved similar to [BGR84, 1.4.1/3] with slight modifications. Of course, it suffices to show that $A\{r^{-1}T\}$ is complete for any non-archimedean Banach ring A and $r \in \mathbb{R}^n_{>0}$.
Let $(f_i)_{i\in\mathbb{N}} = (\sum_{nu=0}^{\infty} a_{i\nu}T^{\nu})$ be a Cauchy sequence in $A\{r^{-1}T\}$. For fixed ν, we have $|a_{i+1\nu} - a_{i\nu}|r^{\nu} \leq \max_{\nu}|a_{i+1\nu} - a_{i\nu}|r^{\nu} = \|f_{i+1}f_i\|_r$. Thus, $(a_{i\nu})_{i\in\mathbb{N}}$ is a Cauchy sequence for each ν. Set $a_{\nu} := \lim_i a_{i\nu}$ and $f := \sum_{\nu}^{\infty} a_{\nu}T^{\nu}$. To see that $f \in A\{r^{-1}T\}$, iteratively throw out finitely many entries of the sequence for each $i \in \mathbb{N}$, such that $||f_j - f_i|| \leq \frac{1}{i}$ for each $i \in \mathbb{N}$ and for all $j >> i$. Hence, from $|a_{j\nu} - a_{i\nu}|r^{\nu} \leq ||f_j - f_i\|_r f \leq \frac{1}{i}$ for all $j >> i$ one deduces that $|a_{\nu} - a_{i\nu}|r^{\nu} \leq \frac{1}{i}$ for all i. ☐

Definition 6.1.14 ([Ber90, 2.1.1])**.** A **k-affinoid algebra** A is a Banach k-algebra, that admits a surjective admissible morphism

$$f : k\{\underline{r}^{-1}\underline{T}\} \to A$$

for some $n \in \mathbb{N}$ and some polyradius $\underline{r} \in \mathbb{R}^n_{>0}$.

A **strictly** k-affinoid algebra is a k-affinoid algebra A, where the surjective admissible morphism is given by $k\{\underline{T}\} \to A$ i.e. $\underline{r} = 1$.

Classical rigid analytic geometry, as developed in [BGR84], focuses on strictly affinoid algebras, which are conventionally termed 'affinoid algebras' in such context.

Note. [Tem11] offers a framework to pass smoothly between general and strictly affinoid algebras by defining a k-affinoid algebra A to be H-strict for some subgroup $|k^*| \subset H \subset \mathbb{R}^{\times}_{>0}$, if the defining $\underline{r}$ lies in H^n.

Lemma 6.1.15 ([Bos14, B.6])**.** *For all $A, B \in$ strict-*Affnd_k*, the tensor product $A\widehat{\otimes}_k B$ is a strictly affinoid k-algebra.*

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[40] This is seen, by iteratively applying the reasoning of Example 6.1.9

Lemma 6.1.16 (Characterisation of free strictly k-affinoid algebras, [BGR84, 6.1.5/4])**.** *Set* $\sqrt{|k^*|} := \{r \in \mathbb{R} \mid \exists n \in \mathbb{N} : \ r^n \in |k|\}$*. Then,* $k\{\underline{r}^{-1}\underline{T}\}$ *is strictly k-affinoid, if and only if* $\underline{r} \in (\sqrt{|k^*|})^n \subset \mathbb{R}^n_{>0}$*.*

Proof. The "only if"-direction, which will not be of further importance in this thesis, is proved by an application of Noether-normalization [BGR84, 6.1.2/1] and [BGR84, 3.1.5/2].

To establish the converse, it suffices to show the statement for single $r \in \sqrt{|k^*|}$, as $k\{\underline{r}^{-1}T\} \cong k\{r_1^{-1}T_1\}\widehat{\otimes}\cdots\widehat{\otimes}k\{r_n^{-1}T_n\}$. By assumption, there is an $s \in \mathbb{N}$ and an $c \in k^*$ such that $r^s = |c|$. Consider the morphism of k-algebras

$$\varphi : k\{T\} \to k\{r^{-1}T\}, \quad T \mapsto cT^s$$

Pick an $f = \sum_i a_i T^i \in k\{r^{-1}T\}$ and set $g_i := \sum_j a_{j\cdot s+i}\, c^{-j} T^j$ for $i = 0, ..., s-1$. Then,

$$f = \sum_{0 \le i < s} T^i (\sum_{0 \le j} a_{j\cdot s+i}\, c^{-j} (cT^s)^j) = \sum_{0 \le i < s} T^i \varphi(g_j)$$

Hence, φ makes $k\{r^{-1}T\}$ a finite $k\{T\}$-module generated by $T^0, ..., T^{s-1}$. Thus

$$\psi\colon k\{T, X_1, ..., X_{s-1}\} \to k\{r^{-1}T\}, \qquad T \mapsto cT^s,\ X_i \mapsto T^i.$$

is well-defined, surjective and continuous[41]. ☐

The following example and proposition demonstrate how certain properties of general k-affinoid algebras can, in many central applications, be recovered from the classical strictly k-affinoid setting.

Example 6.1.17 ([Tem11, Exercise.3.1.1.4])**.** Let $\underline{r} = (r_1, ..., r_n) \in \mathbb{R}^n_{>0}$ be linearly independent over $|k^\times|$. Consider the k-affinoid algebra

$$K_{\underline{r}} := k\{\underline{r}^{-1}\underline{T}\underline{r}\underline{S}\}/(T_1S_1 - 1, ..., T_nS_n - 1).$$

Then $K_{r_i} := k\{r_i^{-1}T_i, r_iS_i\}/(T_iS_i - 1)$ is a field, since each K_{r_i} may be identified with the following subfield the of formal Laurent series[42] $k((T))$

$$\phi\colon K_{r_i} \to k[r_i^{-1}T, r_iT^{-1}] := \{\sum_{j\in\mathbb{Z}} k_jT^j \mid \lim_{|j|\to\infty} |k_j| r_i^j = 0\} \qquad T \mapsto T,\ S \mapsto T^{-1}$$

Here, $k[r_i^{-1}T, r_i, T^{-1}]$ carries the valuation induced by $\|\sum_j k_jT^j\| = \max_j |k_j| r_i^j$, which makes ϕ a bijective isometric homomorphism. Then, by induction

$$K_{\underline{r}} \cong K_{(r_1,...,r_{n-1})}[r_n^{-1}T, r_nT^{-1}] \cong ... \cong K_{(r_1,...,r_{n-1})}\widehat{\otimes}K_{r_n} \cong K_{r_1}\widehat{\otimes}\cdots\widehat{\otimes}K_{r_n}$$

since the r_i are linearly independent. In particular, this makes $K_{\underline{r}}$ a field itself.
If each r_i is not contained in $\sqrt{|k^\times|}$. Then, $K_{\underline{r}}$ is a non-trivially valued field and $|K_{\underline{r}}|$ is generated

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[41] Since $k\{r^{-1}T\}$ is complete, ψ is the unique map from $k\{T, X_1, ..., X_n\}$ induced by the map $k\{T\}[X_1, ..., X_{s-1}] \to k\{r^{-1}T\}$, which is defined in the same way (cf.[BGR84, 6.1.1/4])

[42] The ring $k((T))$ is the Quotient field of the integral domain of formal power series $k[[T]]$.

by $r_1, ..., r_n$ and $|k^\times|$.

Proposition 6.1.18 (Berkovich' Descent, [Ber90, 2.1.2])**.** *Let X be a k-Banach space, and let $r \notin \sqrt{|k^*|}$.*

1) *The canonical map $X \to X\widehat{\otimes}K_r$ is an isometric embedding.*
2) *A sequence of bounded homomorphisms of k-Banach spaces*

$$X \longrightarrow Y \longrightarrow Z$$

is exact and admissible[43] *if and only if the corresponding sequence of K_r-spaces*

$$X\widehat{\otimes}K_r \longrightarrow Y\widehat{\otimes}K_r \longrightarrow Z\widehat{\otimes}K_r$$

is exact and admissible.

Corollary 6.1.19. *For any k-affinoid algebra A, there is an $\underline{r} \in \mathbb{R}^n_{>0}$, such that $A\widehat{\otimes}_k K_{\underline{r}}$ is* ***strictly*** *$K_{\underline{r}}$-affinoid, where $K_{\underline{r}}$ is the field of Def. 6.1.17.*

Corollary 6.1.20. *For any $A, B \in \mathrm{Affnd}_k$ be strict, then tensor product $A\widehat{\otimes}_k B$ is an strictly k-affinoid algebra.*

Proof. Let $\underline{r}'$ and $\underline{r}''$ be the polyradii, such that $A\widehat{\otimes}_k K_{\underline{r}'}$ and $B\widehat{\otimes}_k K_{\underline{r}''}$ are strictly $K_{\underline{r}'}$- resp. $K_{\underline{r}''}$-affinoid. Set $\underline{r} := (\underline{r}', \underline{r}'')$. The proof of Ex. 6.1.17, shows that $A\widehat{\otimes}_k B\widehat{\otimes}_k K_{\underline{r}}$ is strictly $K_{\underline{r}}$-affinoid as it admits an admissible surjective map $K_{\underline{r}}\{\underline{r}^{-1}\underline{T}\}$. As this map is obtained by tensoring the canonical map $c\colon k\{\underline{r}^{-1}\underline{T}\} \to A\widehat{\otimes}_k B$ with $K_{\underline{r}}$, Prop 6.1.18 exhibits c as admissible surjection. ☐

The dimension of an affinoid algebra is an example of a notion, which does not transfer nicely from strict to general k-affinoid algebras. The dimension of a strictly k-affinoid algebra is defined as its Krull-dimension and this invariant is preserved under any extension of the ground field k. However, for general affinoid k-algebras, this is not the case, since $K_r\widehat{\otimes}_k K_r \cong K_r\{T\}$ for $r \notin \sqrt{k}$ (cf.[Tem11, 3.5.1.1.]). This defect is surpassed, by explicitly forcing stability under ground field extensions

Definition 6.1.21. The dimension of an k-affinoid algebra A is defined as the Krull dimension $\dim(A\widehat{\otimes}_k K)$, where K/k is some field extension, such that $A\widehat{\otimes}_k K$ is strictly K-affinoid.

Definition 6.1.22. Let $(A, |\text{-}|_A)$ be a Banach ring. The **spectrum** of A is defined to be the set $\mathcal{M}(A)$ of all semi-valuations on A, which are bounded with respect to $|-|_A$. $\mathcal{M}(A)$ is equipped with the coarsest topology, such that all evaluations $\mathrm{ev}_a : \mathcal{M}(A) \to \mathbb{R}^+, |\text{-}|_x \mapsto |a|_x$ for any $a \in A$ are continuous.

Likewise, the **spectrum** of a k-algebra B **without a fixed valuation** is defined to be the set $\mathcal{M}\,\mathrm{Spec}(B)$ of all multiplicative semi-norms $|\text{-}|\colon B \to \mathbb{R}^+_0$, which extend $|\text{-}|_k$[44], equipped with the weakest topology making all evaluations continuous.

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[43] i.e. all maps in this sequence are admissible

[44] i.e. any $\|\text{-}\|_x \in \mathcal{M}\,\mathrm{Spec}(B)$ is bounded with respect to $|\text{-}|_k$ when restricted to k.

Notation 6.1.23. In case $A = k[X_1, ..., X_n]$ is a free polynomial algebra, denote $\mathbb{A}\mathrm{n}_k^n := \mathcal{M}\operatorname{Spec}(A)$. Thm. 6.3.1 will show, that $\mathbb{A}\mathrm{n}_k^n$ resembles the affine space $\operatorname{Spec}(A) = \mathbb{A}_k^n$.

Lemma 6.1.24. *Assigning a Banach ring $(A, |\text{-}|)$ its spectrum $\mathcal{M}(A)$ yields a functor*

$$\mathcal{M}(\text{-})\colon \mathrm{BaRing}^{op} \to \mathrm{Top}$$

of the category of Banach Rings with bounded homomorphisms to the category of topological spaces.

Notation 6.1.25. $\mathcal{M}(A)$ is called **k-affinoid spectrum**, if $(A, |\text{-}|_A)$ is a k-affinoid algebra. Likewise, we refer $\mathrm{Affnd}_k^{\mathrm{op}}$ as the category of k-affinoid spectra, identifying this category with its essential image under $\mathcal{M}(\text{-})$.

Proof of Lem. 6.1.24. Let $\phi\colon A \to B$ be a bounded homomorphism of Banach Rings. For every $|\text{-}|_x \in \mathcal{M}(B)$ setting

$$f_\phi(|\text{-}|_x) := |\phi(\text{-})|_x = |\text{-}|_x \circ \phi$$

defines a semi-valuation on A, since ϕ is a homomorphisms. Since ϕ is bounded, there are $C, D \in \mathbb{R}_+$ such that

$$|\phi(a)|_x \leq C|\phi(a)|_B \leq C \cdot D|a|_A \ \forall a \in A$$

By definition of f_ϕ, we get a commutative diagram for any $a \in A$

$$\begin{tikzcd}
\mathcal{M}(B) \arrow[r, "f_\phi"] \arrow[dr, "\mathrm{ev}_{\phi(a)}"'] & \mathcal{M}(A) \arrow[d, "\mathrm{ev}_a"] \\
 & \mathbb{R}_+
\end{tikzcd}$$

For any open set $U \subset \mathbb{R}_{>0}$ the diagram implies

$$f_\phi^{-1}\mathrm{ev}_a^{-1}(U) = \mathrm{ev}_{\phi(a)}^{-1}(U)$$

Thus, f_ϕ is continuous, $\mathrm{ev}_{\phi(a)}^{-1}(U)$ is open by Def 6.1.22 and the sets $\mathrm{ev}_a^{-1}(U)$ form a basis for the topology of $\mathcal{M}(A)$. By construction, $f_{\mathrm{id}_A} = id_{\mathcal{M}(A)}$ and $f_{\phi\circ\psi} = f_\psi \circ f_\phi$. ☐

Observation 6.1.26. In the same way, every morphism of k-algebras $\phi\colon A \to B$ induces a continuous map $f\colon \mathcal{M}\operatorname{Spec}(B) \to \mathcal{M}\operatorname{Spec}(A)$ by sending a semi-valuation $\|\text{-}\|\colon B \to \mathbb{R}_{>0}$ to the semi-valuation $f(\|\text{-}\|) := \|\text{-}\| \circ \phi\colon A \to \mathbb{R}_{>0}$.

By the following Observation Banach rings with the same underlying ring and are topologically indistinguishable. Thus, from now on Banach rings with equivalent norms are identified.

Observation 6.1.27. Let A be a ring, which is complete with respect to two equivalent norms $|{-}|_1, |\text{-}|_2\colon A \to R^+$. One uses the constants $c, C > 0$ such that $c| - |_2 \leq | - |_1 \leq C| - |_2$ to show that id_A induces a bounded homomorphism $(A, |{-}|_1) \to (A, |{-}|_2)$ and vice versa. As they are clearly inverse to each other, they give rise to a homeomorphism $\mathcal{M}(A, |\text{-}|_1) \cong \mathcal{M}(A, | - |_2)$

Example 6.1.28. Recall from [Neu92, II.4.8], that for a complete non-archimedean[45] field K with valuation $|\text{-}|_K$ and any algebraic field extension L/K, the field L admits a unique valuation, which extends $|\text{-}|_K$. If L/K is finite, then this is the valuation $|\text{-}|_{N_{K|k}}$ given by the norm of L/K and L is complete wrt. $|\text{-}|_{N_{L|K}}$. In general, this is not the case (cf.[BGR84, A.3.5.1][46]). In the general case, one needs to take the completion $\widehat{L}$ of L wrt. the unique norm induced by L/K. Then, $\widehat{L}$ is algebraic over K, however not necessarily over L.
In any case, $\widehat{L}$ admits only one valuation, which extends $|\text{-}|_K$. Thus, a bounded homomorphism $k \to \widehat{L}$ depends only on choice of underlying field extension $k \hookrightarrow \widehat{L}$. This yields the following isomorphisms for any complete field K and L algebraic over K

$$\mathrm{Hom}_{\mathrm{Affnd}_k^{\mathrm{op}}}(\mathcal{M}(L), \mathcal{M}(K)) \cong \mathrm{Hom}_{\mathrm{Affnd}_k}(K, L) \cong \mathrm{Hom}_{\mathrm{Field}}(K, L).$$

Definition 6.1.29. Let $(A, |\text{-}|_A)$ be a Banach ring and $x := |\text{-}|_x \in \mathcal{M}(A)$ a point. Then, $\ker(|\text{-}|_x)$ is a prime ideal in A. Thus, the integral ring ${}^{A}\!/_{\ker(|\text{-}|_x)}$ inherits the residue semi-norm. Let $\mathcal{H}(x) := Q(\widehat{{}^{A}\!/_{\ker(|\text{-}|_x)}})$ be the completion of its fraction field. $\mathcal{H}(x)$ is called the **completed residue field of** x.

Observation 6.1.30. Every $x \in \mathcal{M}(A)$ induces a canonical morphism $\mathcal{M}(\mathscr{H}(x)) \to \mathcal{M}(A)$ in $\mathrm{Affnd}_k^{\mathrm{op}}$ with image x.

Lemma 6.1.31 ([Ber90, 1.2.1])**.** *For every Banach ring* $(A, \|\text{-}\|$, $\mathcal{M}(A)$ *is a compact Hausdorff space.*

Proof. To verify the Hausdorff property, consider two non-equal $|\text{-}|_x, |\text{-}|_y \in \mathcal{M}(A)$. Then, there is an $f \in A$ and some $r \in \mathbb{R}_{>0}$, such that $|f|_x < r < |f|_y$ (up to renaming). The two subsets of $\mathcal{M}(A)$

$$\{|\text{-}| \in \mathcal{M}(A) \mid r < |f|\}, \qquad \{|\text{-}| \in \mathcal{M}(A) \mid r < |f|\}$$

are each open and each contains either $|\text{-}|_x$ or $|\text{-}|_y$.
Compactness requires an more sophisticated argument. Consider the Gelfand transform

$$A \to G := \prod_{x \in A} \mathscr{H}(x).$$

It is shown in [Ber90, 1.2.3], that $\mathcal{M}(G)$ is compact. This finishes the proof, since the map $\mathcal{M}(G) \to \mathcal{M}(A)$ is surjective. □

Pullbacks in the category $\mathrm{Affnd}_k^{\mathrm{op}}$ are constructed in direct analogy to algebraic geometry, that is by relying on the tensor product.

Lemma 6.1.32. *Any two morphisms* $\mathcal{M}(B) \to \mathcal{M}(A) \leftarrow \mathcal{M}(C)$ *in* $\mathrm{Affnd}_k^{\mathrm{op}}$ *yield a pullback square*

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[45] By Ostrowski's Theorem [Neu92, II.4.2] all of the following holds for archimedean fields as well.

[46] This Lemma even shows, that if L is the algebraic closure of K and L/K is infinite, then L is **never** complete wrt. the unique valuation extending $|\text{-}|_K$.

$$\begin{tikzcd}
\mathcal{M}(B\widehat{\otimes}_A C) \arrow[r] \arrow[d] & \mathcal{M}(C) \arrow[d] \\
\mathcal{M}(B) \arrow[r] & \mathcal{M}(A)
\end{tikzcd}$$

Proof. By [Bos14, B.4] the completed tensor product is the pushout in Affnd_k. $\square$

The spectrum $\mathcal{M}$ also allows for explicit coverings of affine space.

Lemma 6.1.33. *The affine space can be written as $\mathbb{A}\mathrm{n}_k^n = \cup_{\underline{r}\in\mathbb{R}^n_{>0}}\mathcal{M}(k\{\underline{r}^{-1}\underline{T}\})$.*
This specialises to

1) *If the valuation on k is non-trivial, then $\mathbb{A}\mathrm{n}_k^n = \cup_{i\in\mathbb{N}}\mathcal{M}(k\{\underline{c}^{-i}\underline{T}\})$ for a $c \in k^\times$ with $|c| > 1$ and $\underline{c} = (c, .., c)$.*

2) *If the valuation on k is trivial, then this description simplifies to $\mathbb{A}\mathrm{n}_k^n = \mathcal{M}(k\{\underline{T}\})$.*

Proof. Let $\|\text{-}\|_x$ be a semi-valuation on $k[T_1, ..., T_n]$. For any $f = \sum_i a_{\underline{i}}T^{\underline{i}} \in k[\underline{T}]$ the strong triangle inequality implies

$$\|\sum_i a_{\underline{i}}T^{\underline{i}}\|_x \leq \max_{\underline{i}}|a_{\underline{i}}|\|T_1\|_x^{i_1}\cdots\|T_n\|_x^{i_n}. \tag{$\boxed{\mathfrak{A}}$}$$

Set $r_i := \|T_i\|_x$ and $\underline{r} := (r_1, \ldots, r_n)$. Then, $\|\text{-}\|_x$ is bounded with respect to $|\text{-}|_{\underline{r}}$ i.e. $\|\text{-}\|_x \in \mathcal{M}(k\{\underline{r}^{-1}\underline{T}\})$. The additional assertions are proved as follows.

1) Observe that there always exists an $c \in k^\times$, such that $|c| > 1$ by multiplicativity of $|\text{-}|$. Hence, picking a sufficiently large $m \in \mathbb{N}$ such that $c^m \geq \max_{i=1,...,n} r_i$ yields $\|\text{-}\|_x \leq |\text{-}|_{\underline{c}}$ i.e. $\|\text{-}\|_x \in \mathcal{M}(k\{\underline{c}^{-m}\underline{T}\}$.

2) As the valuation on k is trivial, considering $\boxed{\mathfrak{A}}$ yields that, $\|\text{-}\|_x \in \mathcal{M}(\underline{T})$, if $r_i := \|T_i\|_x \leq 1$ for all $i = 1, ..., n$. Otherwise, proceed with an induction on n.
For $n = 1$ and $r_1 > 1$, then Ex. 6.1.11 implies that $k\{r_1^{-1}T\} \cong k[T] \cong k\{T\}$. Therefore, $\mathcal{M}(k\{r_1^{-1}T\} \cong \mathcal{M}(k\{T\})$ by functoriality.
For $n > 1$, we may assume that $r_1, ..., r_{n-1} \leq 1$ by the induction hypothesis. Hence, by the same reasoning

$$k\{\underline{r}^{-1}\underline{T}\} \cong k\{r_1^{-1}T_1, ..., r_{n-1}^{-1}T_{n-1}\}\widehat{\otimes}_k k\{r_n^{-1}T_n\} \overset{6.1.11}{\simeq}\cong k\{r_1^{-1}T_1, ..., r_{n-1}^{-1}T_{n-1}\}\widehat{\otimes}_k k\{T_n\}.$$

Thus, $\mathcal{M}(k\{\underline{r}^{-1}\underline{T}) \cong \mathcal{M}(k\{r_1^{-1}T_1, ..., r_{n-1}^{-1}T_{n-1}\})\times\mathcal{M}(k\{T\})$, which is a subset of $\mathcal{M}(k\{\underline{T}\})$. $\square$

Lemma 6.1.34. *Let A be a k-affinoid algebra and K be a normed, complete field over k. Then, $x \in \mathcal{M}(A)$ is the image of a morphism $f\colon \mathcal{M}(K) \to \mathcal{M}(A)$ in $\mathrm{Affnd}_k^{\mathrm{op}}$ if and only if f factors through $\mathcal{M}(\mathscr{H}(x)) \to \mathcal{M}(A)$.*

Analogously to the case of affine schemes, we would like to assign an k-affinoid spectrum a structure sheaf. To do so, it is ineviteable to fix an analogue of open subschemes. This provides a first problem, as localizations of affinoid algebras are not naturally equipped with an induced norm. Following the generalized definition of rational domains one continues to defines “open subspectra”, called affinoid domains, in this abstract fashion by an universal property.

Definition 6.1.35. Let A be a k-affinoid algebra. A closed subset $V \subset \mathcal{M}(A)$ is an **affinoid domain** of $\mathcal{M}(A)$, if there is a A-affinoid algebra A_V and a morphism of k-affinoid spectra $\psi_V \colon \mathcal{M}(A_V) \to \mathcal{M}(A)$ with image V, such that all morphisms $\mathcal{M}(B) \to \mathcal{M}(A)$ of k-affinoid spectra with image V factor through ψ_V.

An affinoid domain V of $\mathcal{M}(A)$ is **strict**, if V is a strictly k-affinoid algebra.

Observation 6.1.36. Let $V, W \subset \mathcal{M}(A)$ be affinoid domains, then $V \cap W \overset{6.1.32}{\cong} \mathcal{M}(A_V \widehat{\otimes}_A A_W)$ is a k-affinoid domain in $\mathcal{M}(A)$.

Observation 6.1.37. If $\mathcal{M}(A)$ is the spectrum of a strictly k-affinoid algebra, then all affinoid domains $V \subset \mathcal{M}(A)$ are strict.

Note. In [Tem05, Prop.2.1], M.Temkin showed that any monomorphism of k-affinoid spectra induces an isomorphism of completed residue fields at all points of its image. As outlined in [Tem05], this result yields a considerably simpler proof of the classical Geritzen–Grauert Theorem, which states that any affinoid domain is a finite union of rational domains[47]. Thus, the Geritzen-Grauert Theorem identifies Berkovich approach as a direct generalization of classical rigid geometry developed by Tate [Tat71].

Definition 6.1.38. Let $\mathrm{Dom}_{\mathcal{M}(A)}$ be the category whose objects are finite unions of affinoid domains of $\mathcal{M}(A)$ and whose morphisms are inclusions. The elements of $\mathrm{Dom}_{\mathcal{M}(A)}$ are called **k-analytic domains**[48] of $\mathcal{M}(A)$.

This category naturally carries a Grothendieck topology in the following way

Lemma 6.1.39. *A family of morphisms $\{V_i \hookrightarrow V\}_{i \in I}$ in $\mathrm{Dom}_{\mathcal{M}(A)}$ is a covering of V if I is finite and $V = \cup_{i \in I} V_i$. Let $\tau_{\mathcal{M}(A)}$ be the collections of all such coverings. Then $(\mathrm{Dom}_{\mathcal{M}(A)}, \tau_{\mathcal{M}(A)})$ is a Grothendieck presite.*

Proof. Condition 1. and 3. of Def. 2.2.5 are immediately met. Condition 2. follows from 6.1.36. ☐

Lemma 6.1.40 ([Bos14, 4.2.5])**.** *Let $\mathcal{M}(A)$ be a k-affinoid spectrum and let $V \subset \mathcal{M}(A)$ be an affinoid domain. Then, the associated affinoid A-algebra A_V is flat.*

We are ready to show, that the assignment $V \mapsto A_V$ of Def. 6.1.35 provides a structure sheaf $\mathcal{O}_{\mathcal{M}(A)}$ for $\mathcal{M}(A)$. The following generalized version of Tate's Acyclity Theorem even allows to define coherent sheaves over all finite unions of affinoid domains.

Theorem 6.1.41 (Tate's Acyclity Theorem, [Tat71], [Ber90, 2.2.5])**.** *Let $V_1, \ldots, V_n$ be affinoid domains in an affinoid spectrum $\mathcal{M}(A)$ and let M be a finite Banach A-module. Let $V := \cup_{i=1}^n V_i$ be the union of the affinoid domains in $\mathcal{M}(A)$. Denote $V_{i_0,\ldots,i_k} := \cap_{j=0}^k V_{i_j}$ for $0 \leq i_j \leq n$. Then, the (augmented) Čech complex $C^\bullet(V_i, M \otimes A_{-})$ given by*

$$0 \to M \to \prod_{i=1,\ldots,n} M \otimes A_{V_i} \to \prod_{0 \leq i,j \leq n} M \otimes A_{V_{i,j}} \to \ldots \to M \widehat{\otimes} A_{V_{1,\ldots,n}} \to 0$$

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[47] This result even carries over to the H relative setting mentioned in Note 6.1.

[48] In [Ber90, 2.2.5] these subsets are called special subsets.

is exact. Thus, the presheaf

$$\widetilde{M}\colon \mathrm{Dom}^{\mathrm{op}}_{\mathcal{M}(A)} \to A-\mathfrak{Mod},\ V \mapsto \ker(\prod_{i=1,\dots,n} M\otimes A_{V_i} \to \prod_{0\leq i,j\leq n} M\otimes A_{V_{i,j}})$$

is a $\tau_{\mathcal{M}(A)}$-sheaf. In particular, $\mathcal{O}_{\mathcal{M}(A)}\colon V \mapsto A_V$ is a $\tau_{\mathcal{M}(A)}$-sheaf and every finite Banach A-module yields a coherent $\mathcal{O}_{\mathcal{M}(A)}$-module.

Proof. To keep within the scope of this thesis, we will restrict ourselves to reducing the statement Thm. 6.1.41 to the classical Acyclity Theorem of Tate as found in [BGR84, 8.2.1/1], where only strictly affinoid algebras are considered.

Following [BGR84, 8.2.1/5], one reduces to the case $M = A$, first. In case M is a free A-module this is clear, since the complex in question is a finite direct sum of Čech complexes $C^\bullet(V_i, A_-)$. If M is not free, it admits a surjection π from a finite free A-module. Let

$$0 \longrightarrow \ker(\pi) \longrightarrow A^m \xrightarrow{\pi} M \longrightarrow 0$$

be the associated short exact sequence. Lem. 6.1.40 implies the exactness of the associated sequence of (augmented) Čech complexes

$$0 \longrightarrow C^\bullet(V_i, \ker(\pi)) \longrightarrow C^\bullet(V_i, A^m) \longrightarrow C^\bullet(V_i, M) \longrightarrow 0.$$

Thus, $\check{H}^k(V_i, M) \cong \check{H}^{k+1}(V_i, \ker(\pi))$ for all k, by the associated long exact sequence. By Definition of $C^\bullet(V_i, -)$ one has $\check{H}^l(V_i, \ker(\pi)) = 0 = \check{H}^l(V_i, M)$ for all $l \geq n+1$. Thus, for every $k \leq n$, one deduces inductively that $\check{H}^k(V_i, M) = 0$ from $\check{H}^{k+1}(V_i, \ker(\pi)) = 0$ and the exact sequence

$$0 \longrightarrow \check{H}^\bullet(V_i, \ker(\pi)) \longrightarrow \check{H}^\bullet(V_i, A^m) \longrightarrow \check{H}^\bullet(V_i, M) \longrightarrow 0.$$

In case A is strictly affinoid, Thm. 6.1.41 coincides with [BGR84, 8.2.1/1]. If A is not strict, Cor. 6.1.19 provides a poly radius $\underline{r} \in \mathbb{R}^n_{>0}$ such that $A^{\mathrm{st}} := A\widehat{\otimes}K_{\underline{r}}$ is strictly $K_{\underline{r}}$-affinoid. Then, the fact that $C^\bullet(V_i, A^{\mathrm{st}}_-)$ is exact and admissible implies that $C^\bullet(V_i, A)$ is exact and admissible via Berkovich Descent i.e. Prop. 6.1.18. ☐

By a Theorem of Kiehl [KIE66] the correspondence of finite Banach A-modules to coherent sheaves admits a converse for every strictly k-affinoid algebra A. Kiehl's Theorem is generalized to k-affinoid algebras by applying the procedure used Thm. 6.1.41, providing the following result.

Corollary 6.1.42. *For any k-affinoid spectrum $\mathcal{M}(A)$, the assignment $M \mapsto \widetilde{M}$ induces an isomorphism of categories*

$$\widetilde{(-)}\colon A\text{-Mod} \xrightarrow{\cong} \mathcal{O}_{\mathcal{M}(A)}\text{-Mod}.$$

The naive approach to define k-analytic spaces by taking the affinoid domains as admissible open sets does not result in a desired space. A more sophisticated approach is needed.

In [Ber90], V.G. Berkovich introduced the notions of nets and atlases to define k-analytic spaces. The resulting spaces are later endowed with a Grothendieck topology on finite unions of affinoid domains and are equipped with a structure sheaf, which is independent of the chosen atlases.

The upshot to this approach is that one can define atlases allowing only certain affinoid spectra as chart domains. Such a refined construction is developed in [Ber93], where the chosen affinoid spectra satisfy certain conditions relative to a field extension K/k. In practice, these restrictions are rarely needed. An atlas-free approach to k-analytic spaces is presented in [Tem11], where the relation between affinoid domains and open subsets of the analytic space is formulated in terms of quasi-nets.

Definition 6.1.43. A **quasi-net** η on a topological space X is a set of subsets of X, such that for every point $x \in X$, there are $V_1, .., V_n \in \eta$, such that $\bigcup_{i=1}^n V_i$ is a neighbourhood of X.

Definition 6.1.44. Let X be a locally Hausdorff topological space. A **k-analytic domain** in X is a subset $V \subset X$, given by a union of affinoid spectra $V = \cup_{i \in I} V_i$ and the collection $\{V_i\}_{i \in I}$ forms a quasi-net for V when equipped with the subspace topology of X. Let Dom_X be the category of **analytic domains** of X, where the morphisms are given by inclusions.
Likewise, let Dom_X^c be the category of **compact** analytic domains in X i.e. every $V \in \mathrm{Dom}_X^c$ is a *finite* union of affinoid domains. For any $V \in \mathrm{Dom}_X^{(c)}$, a (finite) collection of $\{V_i \subset V\}_{0 \le i \le n} \subset \mathrm{Dom}_X^{(c)}$ is a covering[49], if $V = \cup_{i=0}^n V_i$ and $\{V_i\}_{0 \le i \le n}$ is a quasi-net for V. We denote the collection of all such coverings by G-τ_X resp. G-τ_X^c. These are commonly referred to as **G-topologies**, a terminological relict from the early days of Grothendieck topologies.

Observation 6.1.45. The collections τ_X and τ_X^c defined above form Grothendieck topologies on Dom_X resp. Dom_X^c.

Definition 6.1.46. Let X be a locally Hausdorff topological space and $V \subset X$ be a k-analytic domain, which is the union of k-affinoid spectra $V = \cup_{i \in I} V_i$. The subsets $V_i \subset X$ are called **k-affinoid domains** of X. A (compact) k-analytic domain $V \subset X$ is **strict**, if it is the (finite) union of strict affinoid domains. In this case, one refers to V as a strictly k-analytic domain.

Finally, we are ready to introduce general k-analytic spaces.

Definition 6.1.47. A **k-analytic space** is a locally Hausdorff space X, which admits an open cover by compact k-analytic domains[50], together with the G-topology τ_X^c and a G-τ_X^c-sheaf $\mathcal{O}_X^c$ of Banach k-algebras.
A k-analytic space X is **good**, if every point has an open affinoid neighbourhood.
Moreover, X is **strict**, if it admits an open covering by strict analytic domains $V \in \mathrm{Dom}_X^c$. In this case, one also says that X is strictly k-analytic space.

Remark 6.1.48. By [Tem11, 4.1.3.4], this definition is equivalent to the one in [Ber93], which relies on atlases.

Remark 6.1.49. Unravelling the definitions, every point in an affinoid space X admits an open neighbourhood, which is a compact k-analytic domain V. Although, the affinoid domains $V_1, ..., V_n$ forming V are in general not open, every point in V admits an open neighbourhood *in* V, which is a finite union of affinoid subdomains of $V_1, ..., V_n$.

Remark 6.1.50. By [Tem11, 4.2.1.4] there are k-analytic spaces, which are not good e.g. a

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[49] In [Tem11], this is called an **admissible** covering

[50] Equivalently, the collection of all k-affinoid domains in X forms a quasi-net for X.

polydisc of polyradius $\underline{r} \in \mathbb{R}^n_{>0}$ for $n > 1$, where the open polydisc of radius $\underline{r}$ has been removed.

Example 6.1.51. By Lem. 6.1.33 the affine space $\mathbb{A}\mathrm{n}^n_k$ is a good, strictly k-analytic space.

A k-analytic space is usually regarded as a ringed space whose G-τ-structure sheaf $\mathcal{O}^G_X$ is the G-τ-sheaf of k-algebras provided by the following Lemma

Lemma 6.1.52 ([Tem11, Ex.4.1.3.6])**.** *The τ^c-sheaf $\mathcal{O}^c_X$ extends uniquely to a G-τ-sheaf $\mathcal{O}^G_X$ given by $\mathcal{O}^G_X(V) \cong \mathrm{Hom}_{\mathrm{Ber}_k}(V, \mathbb{A}\mathrm{n}^1_k)$ for every $V \in \mathrm{Dom}_X$.*

Warning 6.1.53. Although, it is called structure sheaf, $\mathcal{O}^G_X$ does *not* determine X in general (cf.[Tem11, 4.1.3.7]). Moreover, $\mathcal{O}^G_X$ is only a sheaf of k-algebras, not *Banach* k-algebras.

Proof of Lem. 6.1.52. Let $V \in \mathrm{Dom}_X$ be an analytic domain and $\coprod_{i \in I} V_i \to V$ a possibly infinite τ_X-covering by affinoid domains. By Def. 6.1.43, every point $v \in V$ is contained in a open neighbourhood $U_v = \cup_{j=1}^n V_{i_j}$. Observe that $U_v \in \mathrm{Dom}^c_X$, as $V_{i_1}, ..., V_{i_n}$ forms a quasi-net for U_v. In consequence, there is a G-τ_X-covering $\{U_k\}_{k \in K}$ of V by compact analytic domains. Then, we simply define $\mathcal{O}^G_X(V)$ as the equalizer k-algebra

$$\mathcal{O}^G_X(V) := eq(\prod_{k \in K} \mathcal{O}^c_X(V_k) \rightrightarrows \prod_{k,l \in K} \mathcal{O}^c_X(V_k) \otimes \mathcal{O}^c_X(V_l))$$

The so defined presheaf $\mathcal{O}^G_X$ already is a G-τ_X-sheaf, as every G-τ_X-covering admits an refinement by affinoid domains. Notice, that every G-τ_X-sheaf extending $\mathcal{O}^c_X$ evaluates on V to this equalizer, because of the sheaf condition. Thus, $\mathcal{O}^G_X$ is unique. ☐

Definition 6.1.54. Let X be a k-analytic space. By Definition, X admits a basis of compact k-analytic domains. Hence, there is an inclusion of sites

$$\iota \colon (X, \tau_{\mathrm{open}}) \hookrightarrow (X, G\text{-}\tau_X)$$

By Thm. 2.3.23, ι induces a geometric morphism

$$\mathrm{Shv}_{G\text{-}\tau_X}(X) \underset{\iota_*}{\overset{\iota^*}{\leftrightarrows}} \mathrm{Shv}_{\mathrm{open}}(X)$$

The sheaf $\mathcal{O}_X := \iota_*(\mathcal{O}^G_X)$ is called the structure sheaf of $\mathcal{O}_X$.

Definition 6.1.55. A morphism $f \colon Y \to X$ between k-analytic spaces consists of a continuous and G-continuous map $f \colon Y \to X$ and a compatible family of bounded homomorphisms

$$f^{\#}_{U,V} \colon \mathcal{O}^c_X(U) \longrightarrow \mathcal{O}^c_Y(V)$$

for any pair of **compact** k-analytic domains $U \subseteq X$ and $V \subseteq f^{-1}(U)$.

Proposition 6.1.56 ([Tem11, 4.1.4.1])**.** *Assume that $\{X_i\}_{i \in I}$ is a family of k-analytic spaces provided with analytic domains $X_{ij} \hookrightarrow X_i$ and isomorphisms $\phi_{ij} \colon X_{ij} \xrightarrow{\sim} X_{ji}$ that satisfy the usual cocycle compatibility condition on the intersections $X_{ijk} = X_{ij} \cap X_{ik}$. In the following cases they glue to a k-analytic space X covered by domains isomorphic to X_i so that $X_i \cap X_j \xrightarrow{\sim} X_{ij}$.*

1) The domains $X_{ij} \subseteq X_i$ are open.

2) The domains $X_{ij} \subseteq X_i$ are closed and for each i only finitely many domains X_{ij} are non-empty.

Many of the common definitions from Algebraic Geometry carry over nicely to Berkovich Geometry.

Definition 6.1.57 ([Tem11])**.** Let $f\colon Y \to X$ be a morphism of k-analytic spaces, then

- f is an **affinoid morphism**, if for any affinoid domain U of X the preimage $U \times_X Y$ is an affinoid domain of Y.
- f is called a **closed immersion** (resp. **finite**) if there exists an G-τ-covering by affinoid domains $X = \cup_i X_i$ such that $Y_i = X_i \times_X Y$ are k-affinoid and the homomorphisms of Banach algebras $\mathcal{O}_{X_G}(X_i) \to \mathcal{O}_{Y_G}(Y_i)$ are surjective (resp. finite) and admissible. A subset $Z \subset X$ is a closed k-analytic subspace of X, if there is a closed immersion $Y \to X$ with image Z.
- A subset $U \subset X$ is **Zariski open**, if $X \setminus U$ is the image of a closed immersion. $f\colon Y \to X$ is an **open immersion** if it is a homeomorphism to an Zariski open subset of X.
- $f\colon Y \to X$ is **separated** if the diagonal morphism $\Delta_{Y/X}\colon Y \to Y \times_X Y$ is a closed immersion. A k-analytic space is **separated** if so is its morphism to $\mathcal{M}(k)$.
- The **relative interior** $\mathrm{Int}(Y/X)$ of $f\colon Y \to X$ is the set of all points $y \in Y$ such that for any affinoid domain $U \subseteq X$ containing $x = f(y)$ there exists an affinoid domain $V \subseteq f^{-1}(U)$ such that V is a neighbourhood of x in $f^{-1}(U)$ and $x \in \mathrm{Int}(V/U)$. The complement $\partial(Y/X) = Y \setminus \mathrm{Int}(Y/X)$ is called the *relative boundary* and we say that f is **boundaryless** or without boundary if $\partial(Y/X)$ is empty.
- $f\colon Y \to X$ is **proper** if it is boundaryless and compact.
- $f\colon Y \to X$ is **finite étale** if for any affinoid domain $U \subseteq X$ its preimage $V = f^{-1}(U)$ is an affinoid domain and $\mathcal{O}_{X_G}(U) \to \mathcal{O}_{Y_G}(V)$ is finite étale.
- In general, a morphism $f\colon Y \to X$ is **étale** if locally (on Y) it is finite étale. Namely, for any point $y \in Y$ there are open neighbourhoods V of y and U of $f(y)$ such that f restricts to a finite étale morphism $V \to U$.

Other important properties of morphisms in Ber_k solely rely on the underlying topological spaces.

Definition 6.1.58 ([Tem11])**.** Let $f\colon Y \to X$ be a morphism in Ber_k. Then,

- f is **Hausdorff**, if the for any two points $y_1, y_2 \in Y$ with $y_1 \neq y_2$ there are neighbourhoods $U_1, U_2 \subset X$ of $f(y_1)$ and $f(y_2)$ such that $U_1 \cap U_2 = \emptyset$.
- $f\colon Y \to X$ is compact, if the preimage of a compact subset is compact.
- Moreover, $f\colon Y \to X$ is said to be **proper**, if it is compact and boundaryless.

Note. Not only is properness a perfect example exhibiting Berkovich geometry as a combination of algebraic and topological nature, but it highlights the difficulties this approach sometimes boars. Indeed, it was uncertain for over twenty years, whether this notion of properness is stable under composition. The general statement has been proven in [Tem04].

The notion of dimension for k-analytic space is deduced from the local as usual.

Definition 6.1.59 ([Duc07])**.** The dimension of a k-affinoid spectrum $\mathcal{M}(A)$ is that of A. The dimension of a k-analytic space X is the supremum of the dimensions of its k-affinoid domains.

This definition behaves well with (topological) irreducibility and the Zariski-topology on X. Ducros established the usual formulas, for instance:

Lemma 6.1.60 ([Duc07, 1.11])**.** *For any normal k-analytic space X and any Zariski closed subset $Z \subset X$:*

$$\dim(Z) + \operatorname{codim}(Z, X) = \dim(X).$$

Let us spell out a few Observations regarding the newly introduced properties, which will be useful later.

Observation 6.1.61. Closed immersions and finite morphisms are affinoid.

Observation 6.1.62. Let A be an k-affinoid algebra and $k\{\underline{r}^{-1}\underline{T}\} \twoheadrightarrow A$ the defining admissible surjection with polyradius $(\underline{r} \in \mathbb{R}^{+})^{n}$ for some $n \in \mathbb{N}$. Then, $\mathcal{M}(A) \to \mathcal{M}(k\{\underline{r}^{-1}\underline{T}\})$ is a closed immersion. Thus, by Lem. 6.1.33, there is always a closed immersion $\mathcal{M}(A) \to \mathbb{A}\mathrm{n}_k^n$.

Moreover, there is a correspondence between ideal sheaves and closed subspaces, analogously to [Har77, 5.9].

Proposition 6.1.63. *The closed k-analytic subspaces of a **good** k-analytic space X correspond directly to the ideal sheaves of $\mathcal{O}_X^G$. Moreover, all ideal sheaves are coherent.*

Proof. Let X be a good k-analytic space and let $\cup_{i \in I}\mathcal{M}(A_i) = X$ be a cover by open affinoid domains. Suppose $i\colon Z \hookrightarrow X$ a closed immersion. Set $\mathscr{J} := \ker(\mathcal{O}_X \to i_*\mathcal{O}_Z)$. Then, $i^{-1}(V_i)$ are open affinoid domains of Z, say corresponding to affinoid k- algebras B_i Then, $\mathscr{J}|_{\mathcal{M}(A_i)}$ is the kernel of $A_i \twoheadrightarrow B_i$, thus an ideal in A_i. Since A_i is noetherian by Prop. B.0.8 , $\mathscr{J}|_{V_i}$ is a finite A_i-submodule of A_i i.e. a finitely generated ideal in A_i.

On the other hand, let $\mathscr{J}$ be a sheaf of ideals of $\mathcal{O}_X$. Thus, the $J_i := \mathscr{J}|_{\mathcal{M}(A_i)}$ are ideals in A_i, hence closed and finitely generated by Prop. B.0.8. Therefore, equipping ${}^{A_i}\!/_{J_i}$ with the residue norm yields the unique Banach A-module structure of ${}^{A}\!/_{I}$ (see Prop. B.0.9). We may assume that each $\mathcal{M}(A_i)$ intersects only finitely many $\mathcal{M}(A_j)$ for $j \in I$, since $\mathcal{M}(A_i)$ is compact. By Prop. 6.1.56 the $\mathcal{M}({}^{A_i}\!/_{J_i})$ specify a gluing datum and thus produce a k-analytic space Z. Then, the associated morphism $Z \to X$ is a closed immersion, since locally on each $\mathcal{M}(A_i) \hookrightarrow X$ the morphism $\mathcal{O}_X(\mathcal{M}(A_i) = A_i \to {}^{A_i}\!/_{J_i} = \mathcal{O}_Z(\mathcal{M}({}^{A_i}\!/_{J_i})$ is surjective. □

Example 6.1.64. Set $A := k[T_1, ..., T_n]/I$. The canonical map $\phi\colon k[\underline{T}] \twoheadrightarrow A$ induces a closed immersion $\mathcal{M}\operatorname{Spec}(A) \to \mathbb{A}\mathrm{n}_k^n$, which corresponds to the ideal sheaf $I \cdot \mathcal{O}^G_{\mathbb{A}\mathrm{n}_k^n}$.

Indeed, by Lem. 6.1.33 the affine space admits the G-cover $\mathbb{A}\mathrm{n}_k^n = \cup_{\underline{r}\in\mathbb{R}^n_{>0}}\mathcal{M}(k\{\underline{r}^{-1}\underline{T})$. Consider the map of Obs. 6.1.26

$$f_\phi\colon \mathcal{M}\operatorname{Spec}(A) \to \mathbb{A}\mathrm{n}_k^n$$

$\mathrm{im}(f_\phi)$ consists of exactly those semi-valuations on $k[\underline{T}]$, which vanish on I. Thus, the kernel of $\mathcal{O}_{\mathbb{A}\mathrm{n}_k^n} \to f_{\phi *}\mathcal{O}_{\mathcal{M}\operatorname{Spec}(A)}$ is given by the sheaf $I \cdot \mathcal{O}_{\mathbb{A}\mathrm{n}_k^n}$, given by

$$I \cdot \mathcal{O}_{\mathbb{A}\mathrm{n}_k^n}(\mathcal{M}(k\{\underline{r}^{-1}\underline{T})) = Ik\{\underline{r}^{-1}\underline{T}\}.$$

Lemma 6.1.65 ([Tem11, 4.2.4.2])**.** *A map $\mathcal{M}(A) \to \mathcal{M}(B)$ of affinoid spectra is proper if an only if it is finite.*

Observation 6.1.66. If $X \mathrel{\longrightarrow\mkern-22mu\vert\mkern14mu} Y$ is a closed immersion and Y is strictly k-analytic, then X is strictly k-analytic, as well. This is seen by unravelling the Definitions.

Thus, if A is a k-algebra of finite type, then $\mathcal{M}\operatorname{Spec}(A)$ is a strictly k-analytic space by Ex. 6.1.64.

Lemma 6.1.67. *Let $X \to Y$ be a finite map in* Ber$k$. *If X is a strictly k-analytic, then so is Y.*

Proof. Let $\{V_i\}_{i\in I}$ be an open cover of Y by compact k-analytic domains. For each $i \in I$, let $\{\mathcal{M}(B_{ij}\}_{j=1,\dots,n_i}$ be a τ^c_Y-cover of V_i by strictly k-affinoid algebras. Then, there are finite k-affinoid algebras $A_{i1}, ..., A_{in}$, such that $A_{ij} \cong f^{-1}(B_{ij})$ is a finite B_{ij}-module. Therefore each A_{ij} is strict and thus $\{f^{-1}(V_i)\}_i$ is an open covering of X by strictly k-analytic domains. ☐

This property carries over to étale maps, which are locally finite by Definition.

Corollary 6.1.68. *Let $X \to Y$ be an étale morphism in* Ber_k*. If X is strictly k-analytic, then so is Y.*

6.2 The Étale Site

To verify that étale morphisms of k-analytic spaces give rise to a Grothendieck topology, we need to establish several fundamental properties.

Lemma 6.2.1. *1) Étale morphisms of k-analytic spaces are stable under base change.*

2) Compositions of Étale morphisms of k-analytic spaces are étale.

3) Étale morphisms have the cancellation property wrt. unramified morphisms i.e. if

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

are morphisms in Ber_k *where g and $g \circ f$ are étale, then f is étale as well.*

Proof. 1. As étale morphisms are defined locally by affinoids, it suffices to show this statement for affinoid spectra. Given two morphisms in $\mathrm{Affnd}_k^{\mathrm{op}}$

$$\mathcal{M}(C) \xrightarrow{g} \mathcal{M}(A) \xleftarrow{f} \mathcal{M}(B)$$

where B is an étale A-algebra. Their pullback is $\mathcal{M}(C\widehat{\otimes}_A B)$ which is affinoid by Lem. 6.1.20. Moreover, $C\widehat{\otimes}_A B$ is an finitely presented B-algebra, since we may write $B = \frac{A\langle X_1,...,X_m\rangle}{(f_1,...,f_n)}$ for some $m, n \geq 0$ and therefore

$$B\widehat{\otimes}_A C = \frac{A\langle X_1,...,X_m\rangle}{(f_1,...,f_n)}\widehat{\otimes}_A C \cong \frac{C\langle X_1,...,X_m\rangle}{(f_1,...,f_n)}$$

by [Bos14, A.5] and [Bos14, A.10]. Further, let $\phi\colon R \to R_0$ be an admissible morphism of complete normed rings such that $\ker(\phi)^2 = 0$ (i.e. a square zero extension) and suppose we have a commutative square in Ban_k

$$\begin{tikzcd}
C \arrow[r] \arrow[d] & R \arrow[d] \\
B\widehat{\otimes}_A C \arrow[r] & R_0.
\end{tikzcd}$$

If we also consider étale morphism $C \to B \otimes_A C$, then we get a bounded homomorphism of A-algebras σ by the definition of (formally) étale morphisms.

$$\begin{tikzcd}
C \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, "\sigma"'] & R \arrow[d] \\
B \otimes_A C \arrow[r] \arrow[urr, dashed] & B\widehat{\otimes}_A C \arrow[r] & R_0.
\end{tikzcd}$$

However, since R is complete, there exists a map $B\widehat{\otimes}_A C \to$ extending σ.

2) It suffices to check that compositions of étale k-algebras are étale. However, compositions of finitely presented morphisms are obviously finitely presented. Moreover, if $f\colon A \to B$ and $g\colon B \to C$ are formally étale, then for every square zero extension $R_0 \to R$, appearing in a square below, there is a lift $\sigma_f\colon B \to R_0$, which in turn yields a lift σ_g exhibiting $g \circ f$ as formally étale morphism.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"] & R_0 \arrow[dd] \\
B \arrow[d, "g"] \arrow[ur, dashed, "\sigma_f"] & \\
C \arrow[r] \arrow[uur, dashed, "\sigma_g"] & R
\end{tikzcd}$$

3) As both étale and unramified maps are G-local on the target, we may assume that all spaces involved are k-affinoid spectra. In this case the assertion is purely algebraic and follows from the corresponding statements for rings (cf.[GW23, 18.30] or [Sta25, Tag 02GW]). ☐

Furthermore, open immersions are special cases of étale morphism, due to the following Lemma, which is stated without proof.

Lemma 6.2.2 ([Ber93, 3.7.4]). *Étale and – more general – smooth morphisms in* Ber_k *are open.*

Similar to the algebraic case, the fibers of étale morphisms are finite separable extensions.

Lemma 6.2.3. *For any étale morphism $f\colon X \to S$ in Ber_k and any point $s \in S$, the fiber $X_s := X \times_S \mathcal{M}(\mathscr{H}(s))$ is a (possibly infinite) coproduct $\coprod_{i\in I} \mathcal{M}(k_i)$ of finite separable field extensions $k_i/\mathscr{H}(s)$.*

Corollary 6.2.4. *If K is an algebraically closed complete valued field and $X \to \mathcal{M}(K)$ is an étale morphism, then X is isomorphic to a disjoint union of copies of $\mathcal{M}(K)$.*

Étale morphism have been defined G-locally in an completely algebraic fashion. It is no surprise that we may thus continue in similar fashion as in algebraic geometry to establish the étale site.

Lemma 6.2.5. *An étale covering of a k-analytic space X is a family of étale morphisms $\{f_i\colon X_i \to X\}_{i\in I}$ such that $\coprod_{i\in I} X_i \to X$ is surjective. Let $\tau_{an\acute{e}t}$ be the collection of étale coverings. Then, $(\mathrm{Ber}_k, \tau_{an\acute{e}t})$ and $(\mathrm{Affnd}_k, \tau_{an\acute{e}t})$ form sites.*

Proof. It follows from Lem. 6.2.1 that étale covers satisfy the conditions of Def. 2.2.5. ☐

Definition 6.2.6. For any X be a k-analytic space, let $X_{an\acute{e}t}$ be the full subcategory of $\mathrm{Ber}_{k/X}$ spanned by étale morphisms $U \to X$. Set $X_{\mathrm{Affnd}} := X_{an\acute{e}t} \times_{\mathrm{Ber}_k} \mathrm{Affnd}_k$ where $X_{an\acute{e}t} \to \mathrm{Ber}_k$ is the canonical projection.

Corollary 6.2.7. *Endowing these categories above with the étale topology yields presites*

$$(X_{an\acute{e}t}, \tau_{an\acute{e}t}), \quad (X_{\mathrm{Affnd}}, \tau_{an\acute{e}t})$$

which are closed under pullbacks. While $X_{\acute{e}t}$ admits a terminal object, X_{Affnd} does not in general.

Notation 6.2.8. For any k-analytic space X, abbreviate $\mathrm{Shv}_{\mathrm{an\acute{e}t}}(X) := \mathrm{Shv}_{\tau_{\mathrm{an\acute{e}t}}}(X_{an\acute{e}t})$.

Example 6.2.9. Let K be an algebraically closed complete valued field. Proceeding as in Ex. 2.7.6, the global section functor Γ induces an equivalence $\mathrm{Shv}_{an\acute{e}t}(\mathcal{M}(K)) \xrightarrow{\simeq} \mathrm{Ani}$ since every $Z \in \mathcal{M}(K)_{an\acute{e}t}$ is a coproduct of copies of $\mathcal{M}(K)$.

By Lem. 2.6.12, sheaves on good k-analytic spaces are determined by their values on affinoids, as with schemes and affines (cf. Prop. C.4.10).

Proposition 6.2.10. *The inclusions $(\mathrm{Affnd}_k, \tau_{an\acute{e}t}) \hookrightarrow (\mathrm{Ber}_k^{\mathrm{good}} \tau_{an\acute{e}t})$ and $(X_{\mathrm{Affnd}}, \tau_{an\acute{e}t}) \hookrightarrow (X, \tau_{an\acute{e}t})$, for any good Berkovich space X, induce equivalences*

$$\mathrm{Shv}_{an\acute{e}t}(\mathrm{Affnd}_k) \simeq \mathrm{Shv}_{an\acute{e}t}(\mathrm{Ber}_k^{\mathrm{good}}), \quad \mathrm{Shv}_{an\acute{e}t}(X_{\mathrm{Affnd}}) \simeq \mathrm{Shv}_{an\acute{e}t}(X).$$

Proof. It suffices to check the conditions *1.-5.* of Lem. 2.6.12.

1) The inclusions are continuous, since étale coverings by k-affinoid spectra are étale coverings.

2) Let $\{f_i\colon U_i \to \mathcal{M}(A)\}_{i\in I}$ be an étale covering of an affinoid spectrum. Then, for every $x \in \mathcal{M}(A)$ and $y \in f_i(x)$, we can find open neighbourhoods $V_x \subset \mathcal{M}(A)$ and $V_i \subset f_i^{-1}(V_x)$ on which f_i is finite étale. By Definition of Berkovich spaces, there is an open neighbourhood $W_y \subset V_x$, which is a finite union of affinoid spectra $\mathcal{M}(A_1), ..., \mathcal{M}(A_n)$. By the Geritzen-Grauert Theorem, there exists an affinoid k-algebra B such that $\mathcal{M}(B) \cong V_x$ and thus there is a finite étale k-affinoid B-algebra C such that $f^{-1}(W_y) \cong \mathcal{M}(C_y)$, since $f|_{V_i}$

is finite étale. In consequence, $\coprod_{x\in X}\coprod_{y\in f_i^{-1}(\{x\})}\mathcal{M}(C_y)\to V_i$ is a covering of U_i and moreover

$$\coprod_{i\in I}\coprod_{x\in X}\coprod_{y\in f_i^{-1}(\{x\})}\mathcal{M}(C_y)\to\mathcal{M}(A)$$

is an étale covering of $\mathcal{M}(A)$. This proves, that both inclusions are continuous.

3) This is condition is trivial, since two maps $f,g\colon U\to U'\in \mathrm{Affnd}$ or X_{Affnd} which coincide after the inclusion are already equal.

4) This condition follows directly from the fact that $\mathrm{Affnd}_k\hookrightarrow \mathrm{Ber}_k$ is fully faithful.

5) Any good k-analytic space U admits an open covering by affinoid domains. $\square$

6.3 Analytification

Let k be a non-archimedean Banach field. In analogy with complex geometry, every scheme locally of finite type over k gives rise to a k-analytic space in a functorial way. While the construction of the associated k-analytic space is more involved than in the complex analytic case, the resulting objects are remarkably well-behaved. To verify these properties and align with modern treatments of Berkovich geometry a slightly streamlined[51] version of the proof of [Ber90, 3.4.1] is provided below.

Theorem 6.3.1 ([Ber90, 3.4.1])**.** *Let X be a scheme locally of finite type over k. Consider the functor*[52]

$$h_X\colon \mathrm{Ber}_k^{\mathrm{good}}\to \mathrm{Set},\ Y\mapsto \mathrm{Hom}_{LocRinged}(Y,X).$$

Then, there is a ***good, strictly*** *k-analytic space X^{an}, called the analytification of X, and a morphism of ringed spaces $\pi_X\colon X^{\mathrm{an}}\to X$, such that h_X is represented by X^{an}.*

In particular, $\mathrm{Spec}(A)^{\mathrm{an}}=\mathcal{M}\,\mathrm{Spec}(A)$ *for any finite type k-algebra A.*

Before we start with the proof of Thm. 6.3.1, let us note two corollaries of universal property of analytification.

Corollary 6.3.2. *Let K/k be a field extension, then K-points of X and X^{an} coincide i.e.*

$$X(K)=\mathrm{Hom}_{Sch/k}(\mathrm{Spec}(K),X)\cong \mathrm{Hom}_{Ber_k}(\mathcal{M}(K),X^{\mathrm{an}})=X^{\mathrm{an}}(K).$$

Corollary 6.3.3. *The assignment $X\mapsto X^{\mathrm{an}}$ yields a functor*

$$(-)^{\mathrm{an}}\colon \mathrm{Sch}^{\mathrm{lft}}_{/k}\to \mathrm{Ber}_k^{good}.$$

Proof. Let $f\colon X\to Y$ be a morphism in $\mathrm{Sch}^{\mathrm{lft}}_{/k}$. Since Y^{an} represents h_Y, there is a unique morphism f^{an} in Ber_k^{good}, such that the following square commutes

---

[51] The original version in [Ber90] does not treat non-good spaces and develops strictness at a later stage. Moreover, the convenient definition of $\mathcal{M}\,\mathrm{Spec}$ was not available.

[52] The curious reader might convince herself, that this is indeed a functor.

$$\begin{tikzcd}
X^{\mathrm{an}} \arrow[r, dashed, "f^{\mathrm{an}}"] \arrow[d, "\pi_X"] & Y^{\mathrm{an}} \arrow[d, "\pi_Y"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

If $f = id_X$, the uniqueness of f^{an} shows that $id_X^{\mathrm{an}} \cong id_{X^{\mathrm{an}}}$. Likewise, for any composable morphisms f, g in $\mathrm{Sch}_{/k}^{\mathrm{lft}}$, one deduces $(g \circ f)^{\mathrm{an}} \cong g^{\mathrm{an}} \circ f^{\mathrm{an}}$ from uniqueness of the analytified morphism. ☐

The proof is carried out by constructing the space X^{an} explicitly by gluing.

Proof of Thm. 6.3.1/Construction of X^{an}. All schemes in this construction are in $\mathrm{Sch}_k^{\mathrm{lft}}$. Consider $A = k[T_1, ..., T_n]$. We assert that $\mathrm{Spec}(A)^{\mathrm{an}} = \mathbb{A}n_k^n$. Pick an $Y \in \mathrm{Ber}^{\mathrm{good}_k}$. From Lem. 6.1.52 follows

$$\mathrm{Hom}_{\mathrm{Ber}_k}(Y, \mathbb{A}n_k^n) \cong \mathrm{Hom}_{\mathrm{Ber}_k}(Y, \mathbb{A}n_k^1)^n = \Gamma(Y, \mathcal{O}_Y)^n.$$

Viewing $\mathcal{O}_Y$ simply as a sheaf of k-algebras, we further get

$$\Gamma(Y, \mathcal{O}_Y)^n \cong \mathrm{Hom}_k(k[T], \Gamma(Y, \mathcal{O}_Y))^n \cong \mathrm{Hom}_{LocRinged}(Y, \mathrm{Spec}(k[T]))^n \cong \mathrm{Hom}_{LocRinged}(Y, \mathbb{A}_k^n).$$

Here the second to last bijection is commonly known if Y were a scheme. However, this bijection even holds for all locally ringed spaces Y as outlined in [GW20, Prop.3.4]. Thus, $\mathrm{Spec}(\mathbb{A}_k^n)^{\mathrm{an}}$ represents $h_{\mathbb{A}_k^n}$. Next, we prove the following

1. Step: *Let X be a scheme and $U \subset X$ a open subscheme. Suppose we have constructed X^{an}, which represents h_X. Then, we can construct an open subspace. $U^{\mathrm{an}} \subset X^{\mathrm{an}}$, which represents h_U.*
In this situation, set $U^{\mathrm{an}} := \pi_X^{-1}(U)$ and $\pi_U := \pi_X|_{U^{\mathrm{an}}}$. As an open subset of X^{an}, U^{an} inherits the structure of a k-analytic space from X^{an}. Pick an arbitrary $Y \in \mathrm{Ber}_k^{\mathrm{good}}$. Then, any morphism $f\colon Y \to U$ of locally ringed spaces factors as follows

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[dr] & U \arrow[r, hook, "i"] & X \\
& X^{\mathrm{an}} \arrow[ur, "\pi_X"'] &
\end{tikzcd}$$

Here, all arrows are morphisms of locally ringed spaces and $Y \to X^{\mathrm{an}}$ is morphism in Ber_k even. By commutativity of this diagram, the morphism $Y \to X^{\mathrm{an}}$ factors through U^{an}. Thus, $\mathrm{Hom}_{\mathrm{Ber}_k^{good}}(Y, U^{\mathrm{an}}) \xrightarrow{\pi_{U^{\mathrm{an}}*}} \mathrm{Hom}_{\mathrm{LocRinged}}(Y, U)$ is surjective. To prove injectivity, apply the same factorization as above to each of some parallel morphisms $f, g\colon Y \to U$ of locally ringed spaces. Then, one observes that

1) $f = g \Leftrightarrow i \circ f = i \circ g$, since $i\colon U \subset X$ is injective;

2) $i \circ f$ resp. $i \circ g$ coincides with the composite $Y \overset{\tilde{f}}{\underset{\tilde{g}}{\rightrightarrows}} U^{\mathrm{an}} \overset{i^{\mathrm{an}}}{\hookrightarrow} X^{\mathrm{an}} \xrightarrow{\pi_X} X$, by commutativity of the diagram;

3) $\pi_X \circ i^{\mathrm{an}} \circ \tilde{f} = \pi_X \circ i^{\mathrm{an}} \circ \tilde{g} \Leftrightarrow \tilde{f} = \tilde{g}$, since $\mathrm{Hom}_{\mathrm{Ber_k}}(Y, X^{\mathrm{an}}) \cong \mathrm{Hom}_{LocRinged}(Y, X)$ and because i^{an} is injective.

In conclusion, $f = g \Leftrightarrow \tilde{f} = \tilde{g}$, which shows, that U^{an} represents h_U.

2. Step: *Suppose we have constructed* X^{an} *for a scheme* X*, which represents* h_X*. Let* $i\colon Z \mathrel{\longrightarrow\!\!\!\!\!\!+}\ X$ *be a closed immersion corresponding to the ideal sheaf* $\mathscr{J} \subset \mathcal{O}_X$*. Then,* Z^{an} *exists and corresponds to the inverse image ideal sheaf* $\pi_X^*(\mathscr{J}) \cdot \mathcal{O}_{X^{\mathrm{an}}}$[53] *and represents* h_Z*.*
Since X^{an} is assumed to be good, there is an unique closed immersion $Z^{\mathrm{an}} \mathrel{\longrightarrow\!\!\!\!\!\!+}\ X^{\mathrm{an}}$ by Prop. 6.1.63 with structure sheaf $\mathcal{O}_{Z^{\mathrm{an}}} \cong {}^{\mathcal{O}_{X^{\mathrm{an}}}}\!\big/_{\pi_X^*(\mathscr{J}) \cdot \mathcal{O}_{X^{\mathrm{an}}}}$. To retrieve a morphism π_Z, consider the composition

$$\mathscr{J} \subset \mathcal{O}_X \to \pi_{X*}\mathcal{O}_{X^{\mathrm{an}}} \to \pi_{X*}i^{\mathrm{an}}_*\mathcal{O}_{Z^{\mathrm{an}}}.$$

By adjunction, this corresponds to the morphism

$$\pi_X^*\mathscr{J} \to \pi_X^*\mathcal{O}_X \cong \mathcal{O}_{X^{\mathrm{an}}} \to i^{\mathrm{an}}_*\mathcal{O}_{Z^{\mathrm{an}}}.$$

By Definition of $\mathcal{O}_{Z^{\mathrm{an}}}$, $\pi_X^*\mathscr{J}$ is send to 0 under this morphism. In consequence, $\mathscr{J}$ is send to 0 under the previous composite. Thus, there is a morphism π_Z of locally ringed spaces such that the following square commutes

$$\begin{tikzcd}
Z^{\mathrm{an}} \arrow[r, "\pi_Z"] \arrow[d, "i^{\mathrm{an}}"] & Z \arrow[d, "i"] \\
X^{\mathrm{an}} \arrow[r, "\pi_X"] & X
\end{tikzcd}$$

To establish the universal property of π_Z, let $f\colon Y \to Z$ be some morphism of locally ringed spaces with $Y \in \mathrm{Ber}_k^{good}$. If we postcompose with $i\colon Z \to X$, we observe that there has to be a morphism $\tilde{f}\colon Y \to X^{\mathrm{an}}$ in Ber_k such that $i \circ f = \pi_X \circ \tilde{f}$. Consider the composition

$$\mathscr{J} \subset \mathcal{O}_X \to i_*\mathcal{O}_Z \to i_*f_*\mathcal{O}_Y = \pi_{X*}\tilde{f}_*\mathcal{O}_Y.$$

Then, $\mathscr{J}$ is sent to 0 under this composition and therefore, $\pi^*\mathscr{J}$ is sent to 0 under the morphism

$$\pi^*\mathscr{J} \to \pi^*\mathcal{O}_X = \mathcal{O}_{X^{\mathrm{an}}} \to \tilde{f}_*\mathcal{O}_Y$$

which is adjoint to the previous composition. Thus, the $\pi_X \circ \tilde{f}$ factors through $i^{\mathrm{an}}\colon Z^{\mathrm{an}} \to X^{\mathrm{an}}$ and therefore through π_Z by the commutativity of the square above. Thus,
$\mathrm{Hom}_{\mathrm{Ber}_k^{good}}(Y, Z^{\mathrm{an}}) \xrightarrow{\pi_Z} \mathrm{Hom}_{\mathrm{LocRinged}}(Y, Z)$ is surjective. The injectivity of this map is deduced in exactly the same way as in Step 1, of course with Z instead of U.

By Step 2, the analytification of a finite type k-algebra $A \cong {}^{k[T]}\!\big/_I$ corresponds to $I \cdot \mathcal{O}_{\mathbb{A}\mathrm{n}_k^n}$. Thus, Example 6.1.64 shows, that $\mathrm{Spec}(A)^{\mathrm{an}} \cong \mathcal{M}\,\mathrm{Spec}(A)$. Let $\cup_{i \in I}\mathrm{Spec}(A_i) = X$ be an open affine covering by finite type k-algebras. For any $i, j \in I$, let U_{ij} denote the intersection $\mathrm{Spec}(A_i) \cap \mathrm{Spec}(A_j)$ considered as an open subset of $\mathrm{Spec}(A_i)$. By Step 1, there is an open

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[53] i.e. the ideal generated by the image of $\pi_X^*(\mathscr{J})$ under the morphism $\pi_X^*(\mathscr{J}) \to \pi_X^*(\mathcal{O}_X) \cong \mathcal{O}_{X^{\mathrm{an}}}$, which is induced by the inclusion. Equivalently, $\pi_X^*(\mathscr{J} \cdot \mathcal{O}_{X^{\mathrm{an}}}$ is given as the ideal sheaf in $\mathcal{O}_{X^{\mathrm{an}}}$ generated by the image of $\mathscr{J}$ under $\pi_X^{-1}\mathcal{O}_X \to \mathcal{O}_{X^{\mathrm{an}}}$ (see [Har77, p.163])

k-analytic subspace $U_{ij}^{\mathrm{an}} \subset \operatorname{Spec}(A_i)^{\mathrm{an}}$, which represents $h_{U_{ij}}$. Thus, the proof of Cor. 6.3.3, shows that there are isomorphisms $U_{ij}^{\mathrm{an}} \overset{\cong}{\to} U_{ji}^{\mathrm{an}}$ for every $i, j \in I$ and the U_{ij}^{an} satisfy the cocycle condition. In conclusion, $\{\operatorname{Spec}(A_i)^{\mathrm{an}}\}_{i\in I}$ specifies a gluing datum (see Prop. 6.1.56) . Moreover, the resulting, glued k-analytic space $X^{\mathrm{an}} = \operatorname{colim}_I \operatorname{Spec}(A_i)^{\mathrm{an}}$ inherits a unique morphism $X^{\mathrm{an}} \to X$ of locally ringed spaces.

To verify that X^{an} represents h_X, let $f\colon Y \to X$ be any morphism of ringed spaces with $Y \in \mathrm{Ber}_k$. Then, the preimages $f^{-1}(\operatorname{Spec}(A_i)$ and $f^{-1}(U_{ij})$ form a gluing datum for Y as k-analytic space, since the transition isomorphisms and cocycle conditions are preserved. Since $\operatorname{Spec}(A_i)^{\mathrm{an}}$ represents $h_{\operatorname{Spec}(A_i)}$, the morphism f factors as

$$\begin{tikzcd}
Y = \operatorname{colim}_I f^{-1}(\operatorname{Spec}(A_i)) \arrow[rr, "f"] \arrow[dr, "\widetilde{f}"'] & & X \\
& X^{\mathrm{an}} = \operatorname{colim}_I \operatorname{Spec}(A_i)^{\mathrm{an}} \arrow[ur, "\pi_X"'] &
\end{tikzcd}$$

By the universal property of colimits, the arrow $\widetilde{f}$ is unique, since each morphism $f^{-1}(\operatorname{Spec}(A_i) \to \operatorname{Spec}(A_i)^{\mathrm{an}}$ is unique. Thus, X^{an} represents h_X. ☐

By the following example, the K points of a locally finite type k-scheme X coincide with the $\widehat{K}$-points of X^{an}.

Example 6.3.4. For a field K any $|-|_x \in \mathcal{M}\operatorname{Spec}(K)$ is a valuation on K by Obs. 6.1.1. If K/k is an algebraic field extension, then $\operatorname{Spec}(K)^{\mathrm{an}} = \mathcal{M}\operatorname{Spec}(K)$ consists of exactly one point (see [Neu92, II.4.8]). If K/k is finite, then this point is exactly the valuation $|-|_{N_{K|k}}$ given by the norm of K/k and K is complete wrt. $|-|_{N_{K|k}}$. Thus, we have $\mathcal{M}(\widehat{K}) \cong \mathcal{M}\operatorname{Spec}(K)$.
In particular, this implies that for $X \in \mathrm{Sch}_{/k}$ any morphism $\mathcal{M}(K) \to X^{an}$ is induced by a morphism $\operatorname{Spec}(K) \to X$.

Proposition 6.3.5. *If $f : X \to Y$ is an morphism in Sch_k^{lft}. Suppose f has one of following properties*

1) f is an open immersion;

2) f is a closed immersion;

3) f is finite;

4) f is étale;

5) f is proper.

Then, $f^{an} : X^{an} \to Y^{an}$ satisfies the same property (as a morphism in Ber_k).

Proof. *1)* and *2)* follow immediately from the proof of Thm. 6.3.1. The explicit description $\operatorname{Spec}(A)^{\mathrm{an}} = \mathcal{M}\operatorname{Spec}(A)$ implies *3)* and *4)*, since these defined G-τ-local via properties of the underlying algebras. *4)* can also be verified by considering the module of differential forms $\Omega_{B/A}$ for any étale morphism of affinoids $\mathcal{M}(B) \to \mathcal{M}(A)$ (cf.[Ber93, 3.3.11]). The last result is established in [Ber90, 3.4.7] with respect to the original notion of properness. In the present work, the conventions for properness follow [Tem11], where this result is stated as Fact 5.1.2.1. ☐

Corollary 6.3.6. *Analytification gives rise to two morphisms of presites*

$$(\text{-})^{\mathrm{an}}\colon (\mathrm{Sch}^{\mathrm{lft}}_{/k}, \tau_{\mathrm{Zar}}) \to (\mathrm{Ber}_k, \tau_{\mathrm{open}}), \qquad\qquad (\mathrm{Sch}^{\mathrm{lft}/k} \tau_{\acute{e}t}) \to (\mathrm{Ber}_k, \tau_{\acute{e}t}).$$

Lemma 6.3.7. $(\text{-})^{\mathrm{an}}\colon \mathrm{Sch}^{\mathrm{lft}}_{/k} \to \mathrm{Ber}^{\mathrm{good}}_k$ *preserves finite limits and connected components.*

Proof. Observe that $\mathcal{M}\,\mathrm{Spec}(k) \cong \mathcal{M}(k)$, thus $(\text{-})^{\mathrm{an}}$ preserves terminal objects.
Moreover, $\mathcal{M}\,\mathrm{Spec}(-)$ sends products to coproducts, since the semi-valuations on a product are defined independently on the factors.
To deduce that $\mathcal{M}\,\mathrm{Spec}(-)$ sends tensor products to pullbacks, first consider the special case of the tensor product $A \otimes_k B$ for finite type k-algebras $A = k[\underline{X}]/I$ and $B[\underline{Y}]/J$. Since

$$A \otimes_k B \cong k[\underline{X},\, \underline{Y}]/(I[\underline{Y}] \cup J[\underline{X}])$$

, we conclude, that $\mathcal{M}\,\mathrm{Spec}(A \otimes_k B)$ is the unique space associated to the sheaf of ideal $(I[\underline{Y}] \cup J[\underline{X}])\mathcal{O}_{\mathbb{A}\mathrm{n}^{\mathrm{m+n}}_{\mathrm{k}}}$ (cf. 6.1.63). However,

$$(I[\underline{Y}] \cup J[\underline{X}])\mathcal{O}_{\mathbb{A}\mathrm{n}^{\mathrm{m+n}}_{\mathrm{k}}} \cong I\mathcal{O}_{\mathbb{A}\mathrm{n}^{m}} \otimes \mathcal{O}_{\mathbb{A}\mathrm{n}^{m}} \cup \mathcal{O}_{\mathbb{A}\mathrm{n}^{m}} \otimes_k J\mathcal{O}_{\mathbb{A}\mathrm{n}^{n}}.$$

The right side is exactly the ideal sheaf corresponding to $\mathcal{M}\,\mathrm{Spec}(A) \times_{\mathcal{M}(k)} \mathcal{M}\,\mathrm{Spec}(B)$. Continuing with the procedure of the proof that the tensor product $A \otimes_B C$ of finite type k-algebras A, B, C is again a finite type k-algebra, one concludes that
$\mathcal{M}\,\mathrm{Spec}(A \otimes_B C) \cong \mathcal{M}\,\mathrm{Spec}(A) \times_{\mathcal{M}\,\mathrm{Spec}(B)} \mathcal{M}\,\mathrm{Spec}(C)$. ☐

7 Constructible Sheaves, Coherence and Points of Algebraic and Analytic Étale Topoi

7.1 Finitely Constructible Sheaves

In this section we generalize the classical notion of constructible sheaves, as found in [Sta25, Tag 005G], to sheaves with values in anima, both for algebraic and analytic geometry. These sheaves will be our main object of study in the next sections.

Convention 7.1.1. Let k denote either $\mathbb{C}$ or a non-archimedean complete field.

Definition 7.1.2. A subset of a topological space is **retrocompact**, if its intersection with every quasi-compact subset is quasi-compact.

Definition 7.1.3 ([Sta25, Tag 005G])**.** A subset E of a scheme X is **constructible** if it is an element in the boolean algebra generated by the **retrocompact** open subschemes of X and the operations $\cup, \cap$ and complement $(\text{-})^c$ i.e. E is of the form $E = \cup_{i=1}^{N} U_i \cap V_i^c$ for retrocompact opens $V_i, U_i \subset X$ and $n \in \mathbb{N}$.

Definition 7.1.4 ([Duc17, 10.1.1])**.** Analogous to Def. 7.1.3, a subset E of a k-analytic space X is constructible, if it is an element in the boolean algebra generated by **Zariski-open** subsets and operations $\cup, \cap$ and complement (-)^c.

Observation 7.1.5. Let $f\colon X \to Y$ be a morphism of schemes or k-analytic spaces. If $E \subset Y$ is constructible, then so is $f^{-1}(E)$, since open immersions are quasi-compact and stable under pullback.

Observation 7.1.6. Let $E \subset X$ be constructible subset of a scheme locally of finite type over k. Then, $E^{\mathrm{an}} \subset X^{\mathrm{an}}$ is constructible, since (-)$^{\mathrm{an}}$ preserves closed immersions by Lem. 5.0.4 and Prop. 6.3.5 and thus sends open subsets to Zariski-open subsets.

Notice, the following easy Lemma.

Lemma 7.1.7. *Let U be an open subset of X, which is retrocompact if X is a scheme. Then, any constructible subset $E \subset U$ is constructible in X.*

Proof. This is [Sta25, Tag 09YD] in case X is a scheme. Otherwise, consider two Zariski-open subsets $W, V \subset U$. Then $W \cap (U \setminus V) = W \cap (X \setminus V) \subset X$ and since E is a finite union of such subsets, it is constructible as a subset of X. ☐

Remark 7.1.8. Def. 7.1.4 differs from Def. 7.1.3 in the way that the Zariski-open subsets generating the boolean algebra do not have to be retrocompact in general. However, for a quasi-compact k-analytic space X all Zariski-open subsets are retrocompact [Duc17, 10.1.3]. Moreover, Ducros [Duc17] demonstrated that this notion of constructibility is sufficiently well-behaved in the sense that it gives rise to analogues of the essential Theorems involving constructibility in algebraic geometry, such as the Proper Chevalley Theorem [Duc17, 10.1.15] or

Definition 7.1.9 (Alexandroff Topology, [BGH20, 1.1.2])**.** Let P be a poset. The Alexandroff topology is on P by defining a set $U \subset P$ to be open, if it satisfies the condition

$$\forall p \in U \, \forall q \in P : p \leq q \Rightarrow q \in U.$$

A P**-stratification** of a topological space X is a continuous map $f\colon X \to P$. In this case, one says X is a P-stratified space and denotes

$$\begin{aligned} X_{\geq p} &:= f^{-1}(\{q \in P \,|\, q \geq p) \\ X_{\leq p} &:= f^{-1}(\{q \in P \,|\, q \leq p) \\ X_p &:= f^{-1}(\{p\}) \end{aligned}$$

Since $X_{\geq p}$ is open and $X_{\leq p}$ is closed, the subset $X_p = X_{\geq p} \cap X_{\leq p}$ is locally closed.

Definition 7.1.10. Let X be either a scheme or a k-analytic space. A sheaf $F \in \mathrm{Shv}_{\acute{e}t}(X)$ is **finitely constructible**, if for every open subset $U \subset X$ there is a **finite** poset P such that U is P-stratified, each U_p is a **constructible** subset and $F|_{U_p}$ is a **finite locally constant** sheaf for any $p \in P$.

Observation 7.1.11. Let X be a scheme or k-analytic space and let $F \in \mathrm{Shv}_{ét}(X)$ be a (p-coprime) finitely constructible sheaf. Then, $\tau_{\leq k}F$ and $\pi_n(F)$ are finitely constructible for every $k \geq -2$, $n \geq 0$, since $\tau_{\leq k}i^*F$ and $\pi_n(i^*F)$ are finite locally constant for every $i\colon U_p \hookrightarrow X$ by

Observation 7.1.12. Let X be either a scheme or a k-analytic space and let $F \in \mathrm{Shv}_{ét}(X)$ be finitely constructible. Then, the automorphism group $\mathrm{Aut}(F)$ is again finitely constructible by virtue[54] of Cor. 4.4.6.

When restricted to qcqs schemes or compact k-analytic spaces the formulation of Def. 7.1.10 vastly simplifies. I thank S.Wolf for pointing out the relation of last two conditions.

Lemma 7.1.13. *Let X be a qcqs scheme or a compact k-analytic space. For any $F \in \mathrm{Shv}_{ét}(X)$, the following are equivalent:*

1) *F is constructible;*

2) *there exists an open covering $X = \bigcup_i U_i$ such that each $F|_{U_i}$ is constructible;*

3) *there exists a finite partition $X = \coprod X_p$ by constructible subschemes/subspaces such that $F|_{X_p}$ is finite locally constant.*

4) *there exists a finite partition $X = \coprod_p X_p$ by qcqs subschemes/compact subspaces such that $F|_{X_p}$ is finite locally constant.*

5) *there is a finite chain of constructible open subsets*

$$\emptyset = U_n \subsetneq U_{n-1} \subsetneq ... \subsetneq U_0 = X$$

such that $F|_{(U_i \setminus U_{i+1})}$ is locally constant for every $i = 0, ..., n-1$.

Proof. (1) ⇒ (2) is trivial.

(2) ⇒ (3): In case X is a scheme, we may also assume that $\cup_i U_i$ is an affine open cover after possibly replacing each U_i an open affine cover of itself. This is possible because, open subsets of constructible subsets are constructible. In both cases, quasi-compactness provides a finite open cover $X = \bigcup_j V_j$, where each V_j admits a finite stratification $V_j = \coprod_{p_j \in P_j} W_{p_j}$ by constructible subsets $W_{p_j} \subset V_j$. Since each W_{p_j} is a constructible subset of X by Lem. 7.1.7, applying [Sta25, Tag 09Y4] produces a finite stratification $X = \coprod_{q \in Q} X_q$, where each X_q is constructible and each V_{p_j} is a finite union of X_qs. Hence, $F|_{X_q}$ is finite locally constant.

(3) ⇒ (4) is trivial, because constructible subsets are retrocompact.

(4) ⇒ (3): This is proved in [Sta25, Tag 095E].

(3) ⇒ (5): Let $f\colon X \to P$ be a finite stratification, such that each X_p is non-empty and qcqs resp. compact. Define the height of a $p \in P$ by setting

$$h(p) := \max\{k \in \mathbb{N}_0 \mid \exists \text{ chain } p_0 < ... < p_k - 1 < p_k = p\}.$$

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[54] Indeed, if $U = \coprod_{p \in P} U_p \subset X$ is an open subset and P is a stratification as in Def. 7.1.10, then $\mathrm{Aut}(F)|_{U_p}$ is finite locally constant by Cor. 4.4.6.

Let $n \leq |P| < \infty$ be maximal length of a chain in P. Then,

$$h\colon (P, \leq) \to [n+1],\ p \mapsto h(p)$$

is a well-defined map of posets. Setting $U_k := f^{-1}h^{-1}(\{k, ..., n\} = \bigcup_{p \in h^{-1}(\{k,...,n)} X_{\leq p}$ for $k = 0, ..., n+1$ provides a chain of constructible open subsets

$$U_{n+1} = \emptyset \subsetneq U_n ... \subsetneq U_1 \subsetneq U_0 = X.$$

Moreover, $U_k \setminus U_{k+1} = \coprod_{p \in h^{-1}(\{k\})} X_p$. Hence, $F|_{U_k \setminus U_{k+1}}$ is finite locally constant, since each $F|_{X_p}$ is finite locally constant.

(5) ⇒ (1): Given a chain of constructible open sets U_k as in (5) set

$$X_i := \coprod_{k \geq i} U_k \setminus U_{k+1},\ i = 0, ..., n-1$$

Then, $X = \coprod_{i=0,...,n-1} X_i$. In particular, for every open subset $V \subset X$, the locally closed sets $V \cap X_i$ are constructible subsets of V. This is [Sta25, Tag 005J] in case X is a scheme. Otherwise, recall that the collection of constructible sets is the boolean algebra generated by Zariski-open sets. However, if $U \subset X$ is a Zariski-open subset of a k-analytic space, then $V \cap U \subset V$ is Zariski-open, as closed immersions are stable under pullback. Hence, $V = \coprod_{i=0,...,n-1} X_i \cap V$ is a finite stratification, that satisfies the conditions of Def. 7.1.10 ☐

Using Lemma, 7.1.13, one deduces, that geometric morphisms induced by morphisms of the schemes or k-analytic spaces or analytification preserve finitely constructible sheaves.

Lemma 7.1.14. *Let $f\colon X \to Y$ be a morphism of qcqs schemes or compact k-analytic spaces and let $f^*\colon \mathrm{Shv}_{\acute{e}t}(Y) \to \mathrm{Shv}_{\acute{e}t}(X)$ be the associated geometric morphism.*
If $F \in \mathrm{Shv}_{\acute{e}t}(Y)$ is finitely constructible, then so is f^*F.

Lemma 7.1.15. *Let Y be a qcqs scheme locally of finite type over k and $F \in \mathrm{Shv}_{\acute{e}t}(Y)$ be finitely constructible. Then, $F^{\mathrm{an}} \in \mathrm{Shv}_{an\acute{e}t}(Y^{\mathrm{an}})$ is finitely constructible.*

Both Lemmas can be proved using the same method.

Proof of Lem. 7.1.14. By Lemma 7.1.13, there is a finite chain of constructible open subsets

$$\emptyset = V_n \subsetneq V_{n-1} \subsetneq ... \subsetneq V_0 = Y$$

such that $F|_{V_{i+1} \setminus V_i}$ is locally constant for every $i = 0, ..., n-1$. Then,

$$\emptyset = f^{-1}V_n \subsetneq f^{-1}V_{n-1} \subsetneq ... \subsetneq f^{-1}V_0 = X$$

is a chain of constructible open set in X by Lem. 7.1.5 and every $f^*F|_{f^{-1}V_i \setminus f^{-1}V_{i+1}}$ is finite locally constant by as geometric morphism preserve locally constant objects by Obs. 3.3.2. ☐

Proof of Lem 7.1.15. Fix a finite chain of constructible open subsets $V_i \subset Y$ as above. Then,

$$\emptyset = V_n^{\mathrm{an}} \subsetneq V_{n-1}^{\mathrm{an}} \subsetneq \dots \subsetneq V_0^{\mathrm{an}} = Y^{\mathrm{an}}$$

is a chain of constructible open subsets in Y^{an} by Lem. 7.1.6 and every $F^{\mathrm{an}}|_{V_i^{\mathrm{an}} \setminus V_{i+1}^{\mathrm{an}}}$ is finite locally constant by Obs. 3.3.2. ☐

Lemma 7.1.16. *Let X be a qcqs scheme or a compact Hausdorff k-analytic space. Then, every finitely constructible étale sheaf is truncated coherent. Thus, there is an inclusion of full subcategories* $\mathrm{Shv}_{\acute{e}t}(X)^{\mathrm{fc}} \subset \mathrm{Shv}_{\acute{e}t}(X)^{\mathrm{tc}}$

This Lemma is deduced from the following Proposition of [DAGXIII], which we are going to use without proof.

Proposition 7.1.17 ([DAGXIII, 2.3.22])**.** *Suppose $\mathcal{X}$ is a $(n+1)$-coherent ∞-topos for $0 \le n \le \infty$*[55]*. Let $U \in \mathcal{X}$ be a quasi-compact (-1)-truncated object, and let*

$$i^*\colon \mathcal{X} \to \mathcal{X}/U, \qquad j^*\colon \mathcal{X} \to \mathcal{X}_{/U}$$

be geometric morphisms associated to the open and closed immersions.
Then an object $X \in \mathcal{X}$ is n-coherent if and only if i^*X and j^*X are n-coherent objects of $\mathcal{X}/U$ and $\mathcal{X}_{/U}$, respectively.
In particular, the ∞-topoi $\mathcal{X}/U$ and $\mathcal{X}_{/U}$ are n-coherent.

Proof of Lemma. 7.1.16. Let $F \in \mathrm{Shv}_{\acute{e}t}(X)^{\mathrm{fc}}$. By Lemma 7.1.13, there is a finite chain of open subsets

$$\emptyset = U_n \overset{j_n}{\subsetneq} U_{n-1} \overset{j_{n-1}}{\subsetneq} \dots \overset{j_1}{\subsetneq} U_0 = X$$

such that $i_k^* F$ is finite locally constant, when we denote $Z_k := U_k \setminus U_{k+1}$ and $i_k^*\colon Z_k \longrightarrow U_k$. Since $F|_{U_n} \simeq j_{n-1}^*(F|_{U_{n-1}})$ is finite locally constant and $F|_{Z_k} \simeq i_k^*(F|_{U_{n-1}})$ is finite locally constant, they are coherent by Cor. 3.3.3, since $\mathrm{Shv}_{\acute{e}t}(X)$ is coherent by Thm. 7.2.10 . Hence, $F|_{U_{n-1}}$ is coherent. Applying this procedure inductively – first for $n-1$ instead of n, then $n-2$ and so forth– one deduces, that F is coherent. ☐

Lurie proved in [DAGXIII, 2.3.25] that these two notions coincide in the broader setting of spectral algebraic geometry. Since this framework readily extends classical algebraic geometry (cf. [DAGV, §2.5, §2.6]), the following proposition is recovered.

Proposition 7.1.18 ([DAGXIII, 2.3.25])**.** *For any qcqs scheme X, the inclusion of Lem 7.1.16 extends to an equivalence*

$$\mathrm{Shv}_{\acute{e}t}(X)^{\mathrm{fc}} \simeq \mathrm{Shv}_{\acute{e}t}(X)^{\mathrm{tc}}.$$

Remark 7.1.19. It is not clear (at least to the author) whether an analogue of this Proposition holds for Berkovich k-analytic space X, as $\mathrm{Shv}_{an\acute{e}t}(X)$ fails to carry the structure of a spectral algebraic space.

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[55] Recall that a ∞-coherent ∞-topos is exactly a coherent ∞-topos.

Lemma 7.1.20. $\mathrm{Shv}_{\acute{e}t}(X)^{\mathrm{fc}}$ *closed under finite limits for every scheme or k-analytic space X.*

Lastly, let us record a general procedure which allows to reduce equivalences of global sections of constructible sheaves to constant sheaves.

Lemma 7.1.21. *Let $f\colon Y \to X$ be a morphism of qcqs schemes or compact k-analytic spaces. Let $F \in \mathrm{Shv}_{\acute{e}t}(X)$ be a constructible sheaf and let*

$$\emptyset = U_n \subset U_{n-1} \subset \ldots \subset U_1 = X$$

a finite chain of constructible open subsets, such that for $Z_l := U_l \setminus U_{l+1}$ the restriction $F|_{Z_l}$ is locally constant and f^* induces an equivalence $\Gamma_{\mathrm{Shv}_{\acute{e}t}(Z_l)}(F|_{Z_l}) \to \Gamma_{\mathrm{Shv}_{\acute{e}t}(f^{-1}Z_l)}(f^*F|_{f^{-1}Z_l})$. Then,

$$\Gamma_{\mathrm{Shv}_{\acute{e}t}(X)}(F) \simeq \Gamma_{\mathrm{Shv}_{\acute{e}t}(Y)}(f^*F).$$

Lemma 7.1.22. *Let X be a qcqs scheme locally of finite type over k and $F \in \mathrm{Shv}_{\acute{e}t}(X)$. Suppose there is a finite chain of constructible open subsets $U_l \subset X$, $l = 0, \ldots, n$ as above, such that $Z_l := U_l \setminus U_{l+1}$ the restriction $F|_{Z_l}$ is locally constant and there is an equivalence $\Gamma_{\mathrm{Shv}_{\acute{e}t}(Z_l)}(F|_{Z_l}) \to \Gamma_{\mathrm{Shv}_{\acute{e}t}(Z_l^{\mathrm{an}})}(F^{\mathrm{an}}|_{Z_l^{\mathrm{an}}})$. Then,*

$$\Gamma_{\mathrm{Shv}_{\acute{e}t}(X)}(F) \simeq \Gamma_{\mathrm{Shv}_{\acute{e}t}(X)}(F^{\mathrm{an}}).$$

Using Lem. 3.3.4, it is possible to reduce the assertion of this Lemma to constant sheaves as well.

Corollary 7.1.23. *Suppose that, in the notation of Lem. 7.1.21, for all constant sheaves G, which locally make up the locally constant sheaves $F|_{Z_l}$, there are equivalences $\Gamma(G) \simeq \Gamma(f^*G)$, then*

$$\Gamma_{\mathrm{Shv}_{\acute{e}t}(X)}(F) \simeq \Gamma_{\mathrm{Shv}_{\acute{e}t}(Y)}(f^*F).$$

Likewise, in notation of Lem. 7.1.22, the existence of equivalences $\Gamma G \simeq \Gamma G^{\mathrm{an}}$, for all constant sheaves which together form $F|_{Z_l}$ implies

$$\Gamma_{\mathrm{Shv}_{\acute{e}t}(X)}(F) \simeq \Gamma_{\mathrm{Shv}_{\acute{e}t}(X^{\mathrm{an}})}(F^{\mathrm{an}}).$$

Proof of Lem. 7.1.21. Denote the chain of constructible subsets $f^{-1}(U_l) \subset Y$ by

$$\emptyset = V_n \subset V_{n-1} \subset, \ldots, \subset V_0 = Y$$

and set $W_l := V_l \setminus V_{l+1}$. By assumption, $\Gamma_{U_{n-1}}(F|_{U_{n-1}}) \simeq \Gamma_{V_{n-1}}(f^*F|_{V_{n-1}})$. Proceeding, inductively, we may suppose that $\Gamma_{U_l}(F|_{U_l}) \simeq \Gamma_{V_l}(f^*F|_{V_l})$ for some $l \leq n-1$. Abbreviate $F := F|_{U_l-1}$. Let $j\colon U_l \to U_{l-1}$, $i\colon Z_{l-1} \to U_{l-1}$ and $j'\colon V_l \to V_{l-1}$, $i'\colon W_{l-1} \to V_l$ denote the inclusions and consider the associated recollements (2.5.4):

$$\mathrm{Shv}_{\acute{e}t}(U_{l-1})_{/y(U_l)} \simeq \mathrm{Shv}_{\acute{e}t}(U_l) \xrightarrow{j_*} \mathrm{Shv}_{\acute{e}t}(U_{l-1}) \xleftarrow{i_*} \mathrm{Shv}_{\acute{e}t}(Z_{l-1}) \simeq \mathrm{Shv}_{\acute{e}t}(U_{l-1})/y(U_l)$$
$$\text{and } \mathrm{Shv}_{\acute{e}t}(V_l) \xrightarrow{j'_*} \mathrm{Shv}_{\acute{e}t}(V_{l-1}) \xleftarrow{i'_*} \mathrm{Shv}_{\acute{e}t}(W_{l-1})$$

fiber sequence of Prop. 2.5.6 induces a map of fiber sequences in Ani

$$\begin{tikzcd}
\Gamma(j_!F|_{U_l}) \arrow[r] \arrow[d] & \Gamma(F) \arrow[r] \arrow[d] & \Gamma(i_*F|_{Z_{l-1}}) \arrow[d] \\
\Gamma(j'_!f^*F|_{V_l}) \arrow[r] & \Gamma(f^*F) \arrow[r] & \Gamma(i'_*f^*F|_{W_{l-1}})
\end{tikzcd}$$

By assumption the right vertical arrow is an equivalence, since $i^* \dashv i_*$ and $i'^* \dashv i'_*$ imply

$$\Gamma_{\mathrm{Shv}_{\acute{e}t}(U_{l-1})}(i_*F|_{Z_{l-1}}) \simeq \Gamma_{\mathrm{Shv}_{\acute{e}t}(Z_{l-1})}(F|_{Z_{l-1}}) \simeq \Gamma_{\mathrm{Shv}_{\acute{e}t}(W_{l-1})}(f^*F|_{W_{l-1}}) \simeq \Gamma_{\mathrm{Shv}_{\acute{e}t}(V_{l-1})}(i'_*f^*F|_{W_{l-1}}).$$

The left vertical map is an equivalence as well, since $j_!$ and $j'_!$ are fully faithful and thus the hypothesis implies

$$\Gamma_{\mathrm{Shv}_{\acute{e}t}(U_{l-1})(j_!F|_{U_l})} \simeq \Gamma_{\mathrm{Shv}_{\acute{e}t}(U_l)}(F|_{U_l}) \simeq \Gamma_{\mathrm{Shv}_{\acute{e}t}(V_l)}(f^*F|_{V_l}) \simeq \Gamma_{\mathrm{Shv}_{\acute{e}t}(V_{l-1})}(j'_!f^*F|_{V_l}).$$

Consequently, $\Gamma_{\mathrm{Shv}_{\acute{e}t}(U_{l-1})}(F) \simeq \Gamma_{\mathrm{Shv}_{\acute{e}t}(V_{l-1})}(f^*F)$ by the associated long exact sequences. ☐

Let G be a finitely constructible group object in the ∞-topoi associated a scheme or k-analytic space. In view of the classification 4.2.8, the Cor. 7.1.23 implies that principle ∞-G-bundles in are uniquely determined by the principal ∞-bundles with constant action group. Furthermore, if G is discrete, then the principal ∞-bundles are even determined by the principal G-bundles – also called G-torsors – in the category of schemes or analytic spaces themselves. The verification of this assertion relies on the sufficient existence of quotients in these categories.

Theorem 7.1.24 ([SGA1, V.1], [Mil17])**.** *Let G be a finite group and $\underline{G}$ the associated group scheme. For every affine scheme X and any action $a\colon X \times \underline{G} \to X$, the quotient Q of a exists and is affine. Moreover, $X \to Q$ is a surjective integral map.*

Lemma 7.1.25 ([BGR84, 6.3.3/3][Ber90, 2.1.14])**.** *Let G be a finite subgroup of automorphisms of a k-affinoid algebra A. Then, the algebra of G-invariant elements A^G is k-affinoid and A is a finite A^G-algebra.*
Thus, $\mathcal{M}(A^G)$ is the quotient of the action $\mathcal{M}(A) \times G \to \mathcal{M}(A)$ and $\mathcal{M}(A) \twoheadrightarrow \mathcal{M}(A^G)$ is finite.

Proposition 7.1.26. *Let X be a scheme or a complex analytic space or a good k-analytic space. Let $C \in \{\mathrm{Sch}, \mathrm{Ber}_k^{\mathrm{good}}\}$ Let G be a finite group and $\underline{G} \in \mathcal{G}\mathrm{rp}(\mathrm{Shv}_{\acute{e}t}(C))$ the corresponding constant discrete sheaf. Then, every $\underline{G}$-principal bundle over X in $\mathrm{Shv}_{\acute{e}t}(C)$ is represented by a principal G-bundle $Y \in C$ over X.*

Proof. Let $P//\underline{G}_\bullet\colon \mathrm{N}(\Delta^{\mathrm{op}}) \to \mathrm{Shv}_{\acute{e}t}(C)$ be a principal ∞-bundle with quotient $X \simeq \mathrm{colim}_{\mathrm{N}(\Delta^{\mathrm{op}})} P//\underline{G}_\bullet$. By Definition every scheme or (good) k-analytic space is a gluing – i.e. a particular colimit – of affine schemes, affine varieties or affinoid domains[56]. Thus, there is a diagram $A_\bullet\colon I \to C$, where each A_i is affine or affinoid, such that

$$P \simeq \mathrm{colim}_{i \in I} L_{\acute{e}t}y(A_i)$$

---

[56] See also Prop. 6.2.10 and Prop. C.4.10.

where $L_{ét}$ denotes the canonical localization from presheaves to sheaves. Since colimits are universal, the augmented simplicial object $P//\underline{G}_\bullet$ can be written as

$$\begin{tikzcd}
\vdots & & \vdots & & \vdots & \\
P \times \underline{G} \times \underline{G} \times \underline{G} \arrow[d, shift left=4.5pt] \arrow[d, shift left=1.5pt] \arrow[d, shift right=1.5pt] \arrow[d, shift right=4.5pt] & \simeq & \operatorname*{colim}_{i\in I} Ly(A_i \times G \times G \times G) \arrow[d, shift left=4.5pt] \arrow[d, shift left=1.5pt] \arrow[d, shift right=1.5pt] \arrow[d, shift right=4.5pt] & & Ly(A_i \times G \times G \times G) \arrow[d, shift left=4.5pt] \arrow[d, shift left=1.5pt] \arrow[d, shift right=1.5pt] \arrow[d, shift right=4.5pt] & \\
P \times \underline{G} \times \underline{G} \arrow[d, shift left=3pt] \arrow[d] \arrow[d, shift right=3pt] & \simeq & \operatorname*{colim}_{i\in I} Ly(A_i \times G \times G) \arrow[d, shift left=3pt] \arrow[d] \arrow[d, shift right=3pt] & & Ly(A_i \times G \times G) \arrow[d, shift left=3pt] \arrow[d] \arrow[d, shift right=3pt] & \\
P \times \underline{G} \arrow[d, shift left=1.5pt] \arrow[d, shift right=1.5pt] & \simeq & \operatorname*{colim}_{i\in I} Ly(A_i \times G) \arrow[d, shift left=1.5pt] \arrow[d, shift right=1.5pt] & \simeq \operatorname*{colim}_{i\in I} \Bigg(& Ly(A_i \times G) \arrow[d, shift left=1.5pt] \arrow[d, shift right=1.5pt] & \Bigg) \\
P \arrow[d, two heads] & \simeq & \operatorname*{colim}_{i\in I} Ly(A_i) \arrow[d, two heads] & & Ly(A_i) \arrow[d, two heads] & \\
Ly(X) & \simeq & \operatorname*{colim}_{[n]\in \mathrm{N}(\Delta^{\mathrm{op}})} \operatorname*{colim}_{i\in I} Ly(A_i \times G^{\times^n}) & \simeq & \operatorname*{colim}_{i\in I} \operatorname*{colim}_{[n]\in \mathrm{N}(\Delta^{\mathrm{op}})} Ly(A_i \times G^{\times^n}) &
\end{tikzcd}$$

Or, as equation: $P//\underline{G}_\bullet = P \times G^{\times^\bullet} \simeq \mathrm{colim}_{i\in I}\, Ly(A_i) \times \underline{G}^{\times^\bullet} \simeq \mathrm{colim}_{i\in I}\, Ly(A_i \times G^{\times^\bullet})$. Consider the simplicial objects $A_i//G_\bullet := A_i \times G^{\times^\bullet}$ on the right side, which are formed in C. By the statements above the quotient $Q_i := \mathrm{colim}_{[n]\in\Delta^{\mathrm{op}}} A_i//G_n$ exists in C, since G is a finite group. Therefore, the last row implies $X \simeq \mathrm{colim}_{i\in I}\, Q_i$. Hence, the affine scheme resp. affinoid domains Q_i glue together to the scheme resp. space X by the data specified the diagram $p\colon I \to C$. Consequently, $p\colon I \to C$ constitutes gluing data and thus, $Y := \mathrm{colim}_{i\in I}\, A_i$ exists in C and is a scheme or k-analytic space. Since L and y preserve colimits, this implies $P \simeq Ly(Y)$ and the principal $\underline{G}$-bundle structure of P exhibits Y as principal G-bundle over X. $\square$

7.2 Coherent ∞-Topoi in Algebraic and Analytic Geometry

For future reference, we list sufficient conditions for recently encountered ∞-sheaves to be coherent. A convenient way to do this is to determine, when objects yield finitary sites in each situation, since finitary sites yield (locally) coherent ∞-topoi by Prop. 2.9.5.

Example 7.2.1. The topology of a complex analytic space X admits a basis by compact Hausdorff spaces (closed disks). The subsite $(\mathrm{Comp}(X), \tau_{\mathrm{open}})$ of $(X, \tau_{\mathrm{open}})$, spanned by these compact Hausdorff spaces is finitary (see Obs. 2.9.4). Moreover, if X is compact, then $(\mathrm{Comp}(X), \tau_{\mathrm{open}})$. contains the terminal object X. Likewise, the site X_{Zar} admits a finitary subsite, if X is quasi-separated. Thus, $\mathrm{Shv}_{\mathrm{open}(X)}$ and $\mathrm{Shv}_{\mathrm{Zar}}(X)$ are locally coherent by Prop. 2.9.5, if X is a complex analytic space or quasi-separated scheme. These ∞-topoi are coherent if X is (quasi)-compact.

To treat algebraic and Berkovich analytic geometry simultaneously, we fix the following notation.

Convention 7.2.2. For the remaining subsection, X denotes either a scheme or a k-analytic space for some non-archimedean field k. Moreover, C denotes either the category of schemes or Ber_k and $\tau_{\acute{e}t}$ stands for both $\tau_{\acute{e}t}$ and $\tau_{an\acute{e}t}$.

The pending discussion in this section primarily concerns the small étale site. However, the preceding example remains relevant in view of the following observation.

Observation 7.2.3. Since étale morphisms are open (cf. Cor 6.2.2 and Cor. C.4.7), a family of étale maps $\{f_i \colon X_i \to X\}$ is a covering if and only if X is covered by the open subsets $\{f_i(X_i) \subset X\}$. Consequently, every étale cover of X admits a finite subcover if and only if X is quasi-compact. In sum, $X_{\acute{e}t}$ is finitary if and only if every $Y \in C$ étale over X is quasi-compact.

However, not every scheme or k-analytic space, which is étale over a quasi-compact X has to be quasi-compact[57]. Since affinoid spectra are compact and Hausdorff (cf. Lem. 6.1.31) and affine schemes are quasi-compact and separated (cf. [Har77, II.Ex.2.13], [FOAG, Ex.3.5C] and [Sta25, Tag 01KI]) they provide a sufficient supply of qcqs subspaces.

Proposition 7.2.4. *For any scheme or good k-analytic space X,* $\mathrm{Shv}_{\acute{e}t}(X)$ *is locally coherent.*

Proof. Recall from Prop. 6.2.10 that $\mathrm{Shv}_{\acute{e}t}(X) \simeq \mathrm{Shv}_{\acute{e}t}(X_{\mathrm{Aff}})$. Thus, in order to show that $\mathrm{Shv}_{\acute{e}t}(X)$ is coherent it suffices to prove that the site $(X_{\mathrm{Aff}}, \tau_{\acute{e}t})$ of affine schemes étale over X is finitary. As argued in Obs. 7.2.3, every étale cover of an affine scheme admits a finite subcover. Using Prop. 2.9.5, we conclude that $\mathrm{Shv}_{\acute{e}t}(X)$ is locally coherent.

Likewise, Prop. 6.2.10 implies that $\mathrm{Shv}_{an\acute{e}t}(X) \simeq \mathrm{Shv}_{an\acute{e}t}(X_{\mathrm{Affnd}})$ and since $(X_{\mathrm{Affnd}, \tau_{an\acute{e}t}})$ is finitary by the argument above, Prop. 2.9.5 shows that this topos is locally coherent. ☐

While $(X_{\mathrm{Aff}}, \tau_{\acute{e}t})$ is finitary, Prop. 2.9.5.3) does not immediately imply that $\mathrm{Shv}_{\acute{e}t}(X)$ is coherent, as X_{Aff} lacks a terminal object if X is non-affine. By Obs. 7.2.3, one cannot define a finitary subsite S of $(X, \tau_{\acute{e}t})$ with a terminal object unless X and all objects of S are quasi-compact. Similar situation are met at k-analytic spaces. This leads to the following definitions.

Definition 7.2.5. For a scheme X, let X_{qc} be the full subcategory of the underlying category of the small étale site $X_{\acute{e}t}$ spanned by the quasi-compact schemes étale over X.

For a fixed k-analytic space X, $\mathrm{Dom}_X^{\mathfrak{c}}$ was defined as the category of compact analytic domains in X i.e finite unions of affinoid domains given by the $\mathcal{O}_X^{\mathfrak{c}}$. If X is compact itself, it is its own compact analytic domain.

Definition 7.2.6. For a k-analytic space X, let $X_{\mathrm{Dom}} \subset X_{an\acute{e}t}$ denote the full category of compact k-analytic spaces étale over X.

Endowing these categories with the étale topology does **not** provide sites, since intersections of quasi-compacts need not be quasi-compact. Surpassing this defect amounts precisely to requiring X to be quasi-separated. (Recall that the topological definition[58] of quasi-separateness coincides with the algebraic definition[59] cf.[FOAG, 11.2.4].)

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[57] If X is not quasi-separated, there even is an open subscheme, which is not quasi-compact (see e.g. [Sta25, Tag 01KO])

[58] i.e. all finite intersections of quasi-compact open subsets are quasi-compact

[59] i.e. X is quasi-separated if the diagonal $X \to X \times X$ is quasi-compact

Lemma 7.2.7. *If X is quasi-separated, then X_{qc} and X_{Dom} form finitary presites with pullbacks when equipped with the étale topology.*

Proof. To prove the statement for X_{qc} and X_{Dom} simultaneously, let X_c stand for either one of these categories. We first show, that X_{c} is closed under finite limits. Suppose $U \to V \leftarrow W$ are morphisms in X_{c} and let $U \times_V W$ be the usual pullback in Ber_k or Sch. Consider the commutative diagram

$$\begin{tikzcd}
U \times_V W \arrow[r] \arrow[d] & W \arrow[d] \arrow[ddr, bend left, "\phi_W"] & \\
U \arrow[r] \arrow[drr, "\phi_U"'] & V \arrow[dr, "\phi_V"] & \\
 & & X
\end{tikzcd}$$

By assumption all maps into X are étale. Thus, all arrows in this diagram are étale, by the cancellation property and stability under pullback (see Lem. ??.2+3). Thus, $\psi\colon U \times_V W \to X$ is étale and in particular open. Therefore, is suffices to prove, that $\mathrm{im}(\psi)$ is quasi-compact. Considering the diagram above, we notice, that $\mathrm{im}(\psi)$ is the open subset of X given by the intersection of quasi-compact open subsets $\phi_U(U) \cap \phi_V(V) \cap \phi_W(W)$. Therefore, $\mathrm{im}(\psi)$ is quasi-compact, as X is assumed to be quasi-separated. Thus, X_c admits all finite limits.

Since X_c is closed under pullbacks, one readily deduces that $(X_{\mathrm{c}}, \tau_{\acute{e}t})$ is a presite from the fact that $X_{\acute{e}t}$ forms a presite. $\square$

Remark 7.2.8. As outlined in [Tem11, 4.1.1.2.iii)], a k-analytic space is Hausdorff if and only if it is quasi-separated. Note that good k-analytic spaces need not be Hausdorff and vice versa (cf.[Tem11, 4.2.3.3]).

Similar to Prop. 6.2.10 étale sheaves on Hausdorff spaces are determined by their values on compacts.

Proposition 7.2.9. *For any k-analytic Hausdorff space X the inclusion $(X_{\mathrm{Dom}}, \tau_{an\acute{e}t}) \hookrightarrow (X, \tau_{an\acute{e}t})$ induces an equivalence*

$$\mathrm{Shv}_{an\acute{e}t}(X_{\mathrm{Dom}}) \simeq \mathrm{Shv}_{an\acute{e}t}(X)$$

Let $\mathit{Hd}\text{-}\mathrm{Ber}_k^{\mathfrak{c}} \subset \mathrm{Ber}_k$ denote the full subcategory of compact Hausdorff spaces. Then, the inclusion of étale sites $(\mathit{Hd}\text{-}\mathrm{Ber}_k^{\mathfrak{c}}, \tau_{an\acute{e}t}) \hookrightarrow (\mathit{Hd}\text{-}\mathrm{Ber}_k, \tau_{an\acute{e}t})$ induces an equivalence

$$\mathrm{Shv}_{an\acute{e}t}(k\text{-}\mathrm{Dom}_{\mathrm{Hd}}^{\mathfrak{c}}) \simeq \mathrm{Shv}_{an\acute{e}t}(\mathit{Hd}\text{-}\mathrm{Ber}_k)$$

Proof. Similiar to Prop. 6.2.10 one checks that the conditions of Lem. 2.6.12 are satisfied, since every k-analytic space admits a covering of compact Hausdorff spaces. $\square$

Proposition 7.2.10. *If X is quasi-compact and quasi-separated, then $\mathrm{Shv}_{\acute{e}t}(X)$ is coherent.*

Remark 7.2.11. For schemes the converse of 7.2.10 holds in greater generality ([SAG, 2.3.4.2]).

Proof of Proposition 7.2.10. If X is an k-analytic space, the statement follows immediately from Lem. 7.2.7, Prop. 7.2.9 and Prop. 2.9.5, since X_{Dom} contains a terminal object.

If X is a scheme, the inclusion $X_{\text{Aff}} \hookrightarrow X_{\acute{e}t}$ factors through X_{qc} since every affine scheme is quasi-compact. A careful examination of the conditions of Lem. 2.6.12 reveals that these are all conditions to the domain not the target. Thus, we get that $X_{\text{Aff}} \hookrightarrow X_{\text{qc}}$ satisfies the conditions of Lem. 2.6.12, as a special case of the fact that $X_{\text{Aff}} \hookrightarrow X_{\acute{e}t}$ satisfy these conditions (cf.Prop. C.4.10). In consequence, we get equivalences

$$\text{Shv}_{\acute{e}t}(X_{qc}) \simeq \text{Shv}_{\acute{e}t}(X_{\text{Aff}}) \simeq \text{Shv}_{\acute{e}t}(X)$$

and since $(X_{\text{qc}}, \tau_{\acute{e}t})$ is finitary and contains a terminal object Prop. 2.9.5 implies that $\text{Shv}_{\acute{e}t}(X)$ is coherent. ☐

Corollary 7.2.12. *By Deligne-Lurie Completeness (Thm. 2.10.11) each of the following ∞-topoi has enough points:*

- $\text{Shv}_{\text{open}}(X)^{\text{hyp}}$ *for any complex analytic space X.*
- $\text{Shv}_{\acute{e}t}(X)^{\text{hyp}}$ *for any scheme X.*
- $\text{Shv}_{an\acute{e}t}(X)^{\text{hyp}}$ *for any good k-analytic space X.*
- $\text{Shv}_{an\acute{e}t}(X)^{\text{hyp}}$ *for any k-analytic Hausdorff space X.*

Thus, equivalences in any of the ∞-topoi above may be checked on their points. To really make use of this principle, it is crucial to classify and understand the nature of the points of the ∞-topoi above. This will be the concern of the next section.

7.3 Points of Étale Topoi

Let $\mathcal{X}$ be the ∞-topos associated with the étale site of a scheme, a good or Hausdorff k-analytic space, or with the site of a complex analytic space endowed with the open topology. As $\mathcal{X}$ has enough points (Cor. 7.2.12), equivalences within $\mathcal{X}$ are detected pointwise. Because $\mathcal{X}$ is induced by a 1-site, it is 1-localic (Prop. 2.6.10), satisfying:

$$\text{Fun}_*(\text{Ani}, \mathcal{X}) \simeq \text{Fun}_*(\text{Set}, \tau_{\leq 0}\mathcal{X}).$$

Consequently, to classify the points in $\mathcal{X}$ it suffices to determine the points of the underlying 1-topos of set-valued sheaves. Naturally, these points coincide with the points of the underlying 1-site $\mathcal{C}$ (Obs. 2.7.3), namely they coincide with the left exact functors $p\colon \mathcal{C} \to \text{Set}$, which send coverings to surjective maps. We start with complex and k-analytic spaces equipped with the open topology.

7.3.1 Sober spaces

Both complex and k-analytic spaces are locally Hausdorff and obey a condition, called soberness, which is also satisfied by Zariski topological spaces. As will be shown below, this property guarantees that the points of the associated topos correspond exactly to the points of the space.

Definition 7.3.1. Let $\mathrm{pt}(X)$ be the topological space given by the set of functors of posets $p\colon \mathrm{Op}(X) \to \{\emptyset, *\}$ such that $p(\emptyset) = \emptyset$ and $p(X) = *$ equipped with the topology, where the open sets are precisely of the form

$$\mathrm{pt}(U) := \{p \in \mathrm{pt}(X) \,|\, p(U) = *\}$$

for an $U \in \mathrm{Op}(X)$.

Observation 7.3.2. As witnessed in Ex. 2.7.5, every $x \in X$ yields a point

$$p_x\colon \mathrm{Op}(X) \xrightarrow{-\times_T t} \mathcal{P}(*;\mathrm{Set}) \xrightarrow{\simeq} \mathrm{Set} \qquad\qquad U \mapsto \begin{cases} * \text{ if } x \in U \\ \emptyset \text{ if } x \notin U \end{cases}$$

Therefore: $$p_x \in \mathrm{pt}(U) \iff x \in U.$$

Thus, the map $\rho\colon X \to \mathrm{pt}(X)$, $x \mapsto p_x$ is continuous.

Note. The functor pt may even be defined for a bigger class of posets called frames. The map ρ arises as a counit of an adjunction between the functor given by $X \mapsto \mathrm{Op}(X)$ and pt. See [MM12, IX] for details.

Remark 7.3.3. In [MM12, p.480, IX.3.(7)] the poset $\mathrm{Op}(X)$ is said to have enough points, if for every $U \subset V \in \mathrm{Op}(X)$

$$\mathrm{pt}(U) = \mathrm{pt}(V) \Rightarrow U = V.$$

If this is the case, the 1-topos $\mathrm{Shv}_{\mathrm{open}}(X;\mathrm{Set})$ also has enough points in the sense of Def. 2.10.8. The converse is also true. Recall from [MM12, III.8.(17)] that the objects of $\mathrm{Op}(X)$ are precisely the subobjects of $1_X \in \mathrm{Shv}_{\mathrm{open}}(X;\mathrm{Set})$. Furthermore, by [MM12, IX.11(5)], a 1-topos $\mathcal{X}$ has enough if and only if for all X, Y

$$X \leq Y \text{ is a subobject } \iff p(X) \subset p(Y) \quad \forall \text{ points } p\colon \mathcal{X} \to \mathrm{Set}.$$

Therefore, if $\mathrm{Shv}_{\mathrm{open}}(X;\mathrm{Set})$ has enough points, then $p(U) \in \{\emptyset, *\}$ for every point $p\colon\ \mathrm{Shv}_{\mathrm{open}}(X;\mathrm{Set}) \to \mathrm{Set}$ and every $U \in \mathrm{Op}(X)$, since $p(X) = *$ and $p(\emptyset) = \emptyset$ as p is exact. In sum, $\mathrm{Op}(X)$ has enough points in the sense of [MM12] if and only if $\mathrm{Shv}_{\mathrm{open}}(X;\mathrm{Set})$ has enough points if and only if $\mathrm{Shv}_{\mathrm{open}}(X)$ has enough points.

Definition 7.3.4. A topological space X is **sober**, if $\rho\colon X \to \mathrm{pt}(X)$ is a homeomorphism.

This Definition might appear non-standard. By [MM12, Prop.IX.3.2] however, it is equivalent to the usual definition via frames or irreducible subsets.

Proposition 7.3.5 ([MM12, IX.3.Cor.4])**.** *A topological space X is is sober if and only if* $\mathrm{Op}(X)$ *has enough points. By Rmk. 7.3.3 this is the case, if and only if* $\mathrm{Shv}_{\mathrm{open}}(X)$ *has enough points.*

Corollary 7.3.6. *Let X be a topological space. If* $\mathrm{Shv}_{\mathrm{open}}(X)$ *has enough points, then* $X \cong \mathrm{pt}(X)$*. In this case, the points of* $\mathrm{Op}(X)$*, which by Thm. 2.3.23 determine the points of* $\mathrm{Shv}_{\mathrm{open}}(X)$*, are precisely the morphisms p_x of Obs. 7.3.2.*

Lastly, we record two well-known examples of sober spaces.

Proposition 7.3.7 ([Nie83, Proposition 3.5])**.** *Topological spaces, which are locally Hausdorff, are sober.*

Proposition 7.3.8. *Any scheme endowed with the Zariski topology is a sober space.*

Proof. In a scheme every irreducible closed set has a unique generic point (cf.[Sta25, Tag 01IS]), which makes it a sober space by [MM12, Prop.IX.3.2]. ☐

7.3.2 Étale Topos of a scheme

For the étale topos of a scheme, the situation is somewhat more subtle. As first outlined in [Ver+72, VIII.7.9], the points of $\mathrm{Shv}_{\acute{e}t}(X;\mathrm{Set})$ for a scheme X correspond precisely to the points of X induced by algebraically closed fields.[60] Since the arguments in [Ver+72] are only sketched and the current version in [Sta25, Tag 04HU] contains a minor gap[61], a detailed argument for this fact is developed below.

Convention 7.3.9. Throughout this subsection, let X denote an arbitrary scheme.

Geometric points and their étale neighbourhoods have been introduced in Ex. 2.7.6. Consult also [Sta25, Tag 03PN] for a more detailed survey.

Notation 7.3.10. Let $\bar{x}\colon \mathrm{Spec}(K) \to X$ be a geometric point. Continuing the notation of Ex. 2.7.6, we denote the induced point of $X_{\acute{e}t}$ by $F_{\bar{x}}\colon X_{\acute{e}t} \to \mathrm{Set}, U \mapsto |U_{\bar{x}}|$.

In basic cases, examined below, the images of $F_{\bar{x}}$ can be expressed in terms of Galois groups.

Lemma 7.3.11. *Let* $\bar{x}\colon \mathrm{Spec}(K) \to X$ *be a geometric point with underlying point* $x \in X$*. Let* $f\colon (U,\bar{u}) \to (X,\bar{x})$ *be an étale neighbourhood,* $f^{-1}(x) = \{u\}$ *for a point* $u \in U$*. Consider the extensions* $\kappa(x) \subset \kappa(u) \subset K$*, where* $\kappa(u)/\kappa(x)$ *is finite separable and let* N *be the normal closure of* $\kappa(u)$ *in* K*. Then,*

$$F_{\bar{x}} \cong |U_{\bar{x}}| \cong \mathrm{Gal}(N/\kappa(x))$$

Proof. By Definition, $N/\kappa(x)$ is a finite Galois subextension of $K/\kappa(x)$. As every extension $K/\kappa(u)$ is separable, the set of geometric points $\mathrm{Spec}(K) \to U$ lying over $u \in U$ corresponds uniquely to

$$\mathrm{Hom}(\kappa(u), K) \cong \mathrm{Hom}(N, K) \cong \mathrm{Gal}(K/N)$$

where $\mathrm{Gal}(K/N)$ is a possibly infinite Galois group. Moreover, the set of geometric points $\bar{u}\colon \mathrm{Spec}(K) \to U$ over $\bar{x}$, i.e. $|U_{\bar{x}}|$, is precisely the set of $\kappa(x)$-algebra morphisms

$$\mathrm{Hom}_{\kappa(x)}(\kappa(u), K) \cong \mathrm{Hom}_{\kappa(x)}(N, K).$$

The main Theorem of (infinite) Galois-Theory (c.f. [Lan02, VI.1.2]) implies that precomposing with $i\colon \kappa(x) \subset \kappa(u)$ induces the commutative diagram

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[60] More precisely, [Ver+72, VIII.7.9] shows that the points correspond to separably closed fields over X.

[61] Let $u\colon X_{\acute{e}t} \to \mathrm{Set}$ be some point of the étale site. The current version of [Sta25, Tag 04HU] lacks an argument showing that $u(U)$ is finite for any connected scheme U étale over X.

$$\begin{tikzcd}
\mathrm{Hom}(N,K) \arrow[r, "i^*"] \arrow[d, phantom, sloped, "\cong"] & \mathrm{Hom}(\kappa(x),K) \arrow[d, phantom, sloped, "\cong"] \\
\mathrm{Gal}(K/N) \arrow[r, phantom, "\leq"] & \mathrm{Gal}(K/\kappa(x))
\end{tikzcd}$$

Here the identity in $\mathrm{Gal}(K/\kappa(x))$ corresponds to the fixed embedding $\bar{x} \in \mathrm{Hom}(\kappa(x), K)$. Taking $\bar{x}$ to be the neutral element in $\mathrm{Hom}(\kappa(x), K)$ one obtains

$$ {}^{\mathrm{Gal}(K/\kappa(x))}\!\big/_{\mathrm{Gal}(K/N)} \cong \mathrm{coker}(i^*)\,\mathrm{Hom}_{\kappa(x)}(N,K)$$

As in finite Galois theory, the left hand side simplifies to $\mathrm{Gal}(N/\kappa(x))$ (c.f.[Lan02, VI.1.11]). In sum, this implies

$$|U_{\bar{x}}| \cong \mathrm{Hom}_{\kappa(x)}(N,K) \cong {}^{\mathrm{Gal}(K/\kappa(x))}\!\big/_{\mathrm{Gal}(K/N)} \cong \mathrm{Gal}(N/\kappa(x)). \qquad \square$$

Lemma 7.3.12. *The assignment $x \mapsto F_{\bar{x}}$ yields an injection*

$$\{\textit{Geometric points of } X\} \hookrightarrow \mathrm{Fun}^*(X_{\acute{e}t}, \mathrm{Set}).$$

Proof. Consider two geometric points $\bar{x}, \bar{y}\colon \mathrm{Spec}(K) \to X$ and suppose that the there is an isomorphism of functors $F_{\bar{x}} \cong F_{\bar{y}}$. Let $x, y \in X$ denote the images of $\bar{x}$ and $\bar{y}$. [Sta25, Tag 02LF] implies that for every finite separable field extension $L/\kappa(x)$ there exists a étale neighbourhood $(U, \bar{u}) \to (X, \bar{x})$ with $\kappa(u) = L$, where $u \in U$ is the image of $\bar{u}\colon \mathrm{Spec}(K) \to U$. Since étale maps are locally quasi-finite ([Sta25, Tag 03WS]), their fibres consist of finitely many isolated points. Thus we may assume that u is the only lying over x (cf.Lem. C.5.5). Since $F_{\bar{y}}(U) \cong F_{\bar{x}}(U)$ there must be exactly one point $v \in U$ lying over y. Let $M/\kappa(x)$ and $N/\kappa(y)$ be the normal closures of $\kappa(u)/\kappa(x)$ and $\kappa(v)/\kappa(y)$. Then, by Lem. 7.3.11, there is an equivalence

$$\mathrm{Gal}(M/\kappa(x)) \cong F_{\bar{x}}(U) \cong F_{\bar{y}}(U) \cong \mathrm{Gal}(N/\kappa(y)).$$

Vice versa, every finite separable extension over $\kappa(y)$ gives rise to a finite separable extension over $\kappa(x)$, with isomorphic Galois group. Since K is the colimit of all finite separable field extensions over $\kappa(x)$ or over $\kappa(y)$ and $\mathrm{Gal}(K/\kappa(x)$ is the associated profinite group, this implies

$$\mathrm{Gal}(K/\kappa(x)) \cong \lim_{L/\kappa(x) \text{finite separable}} \mathrm{Gal}(M_L/\kappa(x)) \cong \lim_{L/\kappa(y)} \mathrm{Gal}(N_L/) \cong \mathrm{Gal}(K/\kappa(y)).$$

Hence, $\kappa(x) = \kappa(y)$. Since open immersions are étale, every open neighbourhood U of x has to contain y and vice versa (otherwise $F_{\bar{x}}(U) \ncong F_{\bar{y}}(U)$). Hence, one point has to be a specialization of the other, but then $\kappa(x) = \kappa(y)$ implies that $x = y$. $\square$

The goal of this subsection is to verify that every point of X is of the form $F_{\bar{x}}$. First we show, that every point admits a natural transformation coming from such a point.

Lemma 7.3.13. *Let $p\colon X_{\acute{e}t} \to \mathrm{Set}$ be a point of $X_{\acute{e}t}$. Then, there is a geometric point $\bar{x}\colon \mathrm{Spec}(K) \to X$ and natural transformation $\phi\colon F_{\bar{x}} \Rightarrow p$, where $F_{\bar{x}}$ is the point induced by $\bar{x}$.*

To establish this assertion the following result is needed.

Lemma 7.3.14 ([Sta25, Tag 03PQ])**.** *Let X be a scheme. For any geometric point $\bar{x}\colon \mathrm{Spec}(K) \to X$, the category $\mathrm{Nb}_{\text{ét}}(\bar{x})$ is cofiltered.*

A proof of Lem. 7.3.14 is given in [Sta25, Tag 03PQ]. Since the same reasoning will be applied in Lem. 7.3.29 in the framework of Berkovich geometry, the proof is omitted here.

Proof of Lem. 7.3.13. Precomposing with the inclusion of the Zariski into the étale site yields

$$X_{\mathrm{Zar}} \hookrightarrow X_{ét} \xrightarrow{p} \mathrm{Set}$$

As X is a sober space (Prop. 7.3.8), there is a point $x \in X$ such that the composition above is isomorphic to a functor $p_x\colon X_{Zar} \to \mathrm{Set}$ as in Def. 7.3.2. Let $f\colon U \to X$ be an étale morphism, then $f(U) \subset X$ is open by Cor. C.4.7. Since $U \to f(U)$ is an étale covering, the induced map $p(U) \to p(f(U))$ is surjective. In conclusion

$$p(U) \neq \emptyset \Leftrightarrow p(f(U)) = p_x(f(U)) \neq \emptyset \Leftrightarrow x \in f(U).$$

Let $\bar{x}\colon \mathrm{Spec}(K) \to \mathrm{Spec}(\kappa(s)) \to X$ be the geometric point lying over x obtained by taking an algebraic closure $K := \overline{\kappa(x)}$ of the residue field $\kappa(x)$. Denote the associated point by

$$F_{\bar{x}}\colon X_{ét} \to \mathrm{Spec}(K)_{ét} \cong \mathrm{Set},\, U \mapsto |U_{\bar{x}}|$$

By Thm. 2.3.23, $F_{\bar{x}}$ and p induce geometric morphisms of topoi

$$G_{\bar{x}}, P\colon\ \mathrm{Shv}_{ét}(X) \to \mathrm{Set}$$

and this association is fully faithful. Therefore, we conclude

$$\begin{aligned}
&\mathrm{Hom}_{\mathrm{Fun}(X_{ét},\mathrm{Set})}(F_{\bar{x}}, p) \cong \mathrm{Hom}_{\mathrm{Fun}(\mathrm{Shv}_{ét}(X),\mathrm{Set}}(G_{\bar{x}}, P) \\
&\overset{2.7.9}{\cong} \mathrm{Hom}_{\mathrm{Fun}(\mathrm{Shv}_{ét}(X),\mathrm{Set})}\Big(\operatorname*{colim}_{(U,\bar{u})\in \mathrm{Nb}_{ét}^{\mathrm{op}}(\bar{x})} \mathrm{Hom}(y(U), -), P\Big) \\
&\cong \lim_{(U,\bar{u})\in \mathrm{Nb}_{ét}(\bar{x})} \mathrm{Hom}_{\mathrm{Fun}(\mathrm{Shv}_{ét}(X),\mathrm{Set})}(\mathrm{Hom}(y(U), -), P) \\
&\cong \lim_{(U,\bar{u})\in \mathrm{Nb}_{ét}(\bar{x})} P(y(U)) \cong \lim_{(U,\bar{u})\in \mathrm{Nb}_{ét}(\bar{x})} p(U)
\end{aligned}$$

For every étale neighbourhood $(U, \bar{u}) \in \mathrm{Nb}_{ét}(\bar{x})$, the set $p(U)$ is non-empty, since x lies in the image of $U \to X$. As $\mathrm{Nb}_{ét}(\bar{x})$ is cofiltered by Lem. 7.3.14, it follows that $\lim_{(U,\bar{u})} p(U) \neq \emptyset$. ☐

To verify that the images of finite étale $(U \to X)X_{ét}$ are finite for any point $p\colon X_{ét} \to \mathrm{Set}$ we will rely on a few facts about Galois categories, which are defined in Def. C.8.1

Remark 7.3.15. Fix a geometric point $\bar{x}\colon \mathrm{Spec}(K) \to X$. Let $\mathrm{F\acute{E}t}_X$ denote the category of schemes finite étale over X. Since finite étale maps are stable under basechange and composition, this category forms a subsite of $X_{ét}$. Suppose X is connected. By [Sta25, Tag 0BNB], the tuple $(\mathrm{F\acute{E}t}_X, F_{\bar{x}})$ forms a Galois category, where $F_{\bar{x}}\colon \mathrm{F\acute{E}t}_X \to \mathrm{Set}$ is the fiber functor induced by $\bar{x}$.

Let $|G := \mathrm{Aut}(F_{\bar{x}})$ be the group of automorphic natural transformations of $F_{\bar{x}}$. By [BS14, 7.2.5] or equivalently [Sta25, Tag 0BNB] there is an equivalence

$$F_{\bar{x}} \colon \mathrm{F\acute{E}t}_X \xrightarrow{\cong} \text{Finite-}G\text{-Set}.$$

This equivalence can be used to show that all points take finite values on $\mathrm{F\acute{E}t}_X$.

Lemma 7.3.16. *Let X be connected. For any point $p \colon X_{\acute{e}t} \to \mathrm{Set}$ and any finite étale $U \to X$ the set $p(U)$ is finite.*

Proof. For any $U \in \mathrm{F\acute{E}t}_X$ set $\mathrm{Aut}_X(U) := \mathrm{Aut}_{\mathrm{F\acute{E}t}_X}(U)$. Let $\bar{x}$ be the geometric point of X and $\phi \colon F_{\bar{x}} \Rightarrow p$ the associated natural transformation provided by Lem. 7.3.13. Let $U \in \mathrm{F\acute{E}t}_X$ be a Galois object, i.e. assume that $\mathrm{Aut}_X(U)$ acts simply transitively on $F_{\bar{x}}(U)$. Thus,

$$F_{\bar{x}}(U) \times \mathrm{Aut}_X(U) \to F_{\bar{x}}(U) \times F_{\bar{x}}(U), \qquad (u, \sigma) \mapsto (u, \sigma(u))$$

is an isomorphism. Since $X \in \mathrm{F\acute{E}t}_X$ is the terminal object, we have $F_{\bar{x}}(X) \cong *$. Therefore,

$$\begin{aligned} F_{\bar{x}}(X) = * &\cong \mathrm{coeq}(F_{\bar{x}}(U) \times \mathrm{Aut}_X(U) \underset{\mathrm{pr}_{F_{\bar{x}}(U)}}{\overset{a}{\rightrightarrows}} F_{\bar{x}}(U)) \\ &\cong \mathrm{coeq}(F_{\bar{x}}(U) \times F_{\bar{x}}(U) \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} F_{\bar{x}}(U)) \end{aligned}$$

where a is the action i.e. $a(u, \sigma) := \sigma(u)$. Naturally, the set $\mathrm{Aut}_X(U)$ acts on U, as well. Since $F_{\bar{x}}$ is an equivalence, we conclude that this action must be principal as well, in the sense that there is the following commutative diagram.

$$\begin{tikzcd}
U \times \mathrm{Aut}_X(U) \arrow[r, shift left, "a"] \arrow[r, shift right, "\mathrm{pr}_U"'] \arrow[d, "\cong"] & U \arrow[r] \arrow[d, "\cong"] & \mathrm{coeq}(a, \mathrm{pr}_U) \arrow[r, phantom, "\cong"] & X \arrow[d, equal] \\
U \times U \arrow[r, shift left, "\mathrm{pr}_1"] \arrow[r, shift right, "\mathrm{pr}_2"'] & U \arrow[r] & \mathrm{coeq}(\mathrm{pr}_1, \mathrm{pr}_2) \arrow[r, phantom, "\cong"] & X
\end{tikzcd}$$

The lower row is expands precisely to the Čech nerve of finite étale surjective[62] map $f \colon U \to X$. Embedding this diagram into $\mathrm{Shv}_{\acute{e}t}(X; \mathrm{Set})$ shows, yields a principal bundle

$$y(U) \times \underline{\mathrm{Aut}_X(U)} \rightrightarrows y(U) \overset{y(f)}{\twoheadrightarrow} y(X)$$

where $y(f)$ is the effective epimorphism associated to f and $\underline{\mathrm{Aut}_X(U)} \in \mathrm{Shv}_{\acute{e}t}(X; \mathrm{Set})$ is the constant sheaf. Let $P \colon \mathrm{Shv}_{\acute{e}t}(X; \mathrm{Set}) \to \mathrm{Set}$ denote the geometric morphism induced by p. Since Set is the initial 1-topos (cf. 2.3.24), P sends $\underline{\mathrm{Aut}_X(U)})$ to the set $\mathrm{Aut}_X(U)$. Moreover, $p(V) = P(y(V))$ for all $V \in X_{\acute{e}t}$. As P preserves principal colimits, finite limits and in particular effective epimorphisms and principal bundles, we get the commutative diagram

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[62] If $f \colon U \to X$ is finite étale, then $f(U)$ is closed and open in X. Therefore, f is surjective if X is connected.

$$\begin{tikzcd}
p(U) \times \mathrm{Aut}_X(U) \arrow[r, shift left, "a"] \arrow[r, shift right, "\mathrm{pr}_U"'] \arrow[d, "\cong"] & p(U) \arrow[r] \arrow[d, "\cong"] & \mathrm{coeq}(a, \mathrm{pr}_U) \arrow[d, "\cong"] & & \\
p(U) \times p(U) \arrow[r, shift left, "\mathrm{pr}_1"] \arrow[r, shift right, "\mathrm{pr}_2"'] & p(U) \arrow[r] & \mathrm{coeq}(\mathrm{pr}_1, \mathrm{pr}_2) \arrow[r, phantom, "\cong"] & p(X) \cong *
\end{tikzcd}$$

Thus, the finite set $\mathrm{Aut}_X(U)$ acts transitive on $p(U)$, since quotient i.e. the coequalizer of this action is a point. Therefore, $p(U)$ must be finite. The vertical isomorphism shows, that this action is free and therefore $p(U) \cong |\,\mathrm{Aut}_X(U)|$.

Take an arbitrary connected $V \in \mathrm{F\acute{E}t}_X$. V admits a map $g\colon U \to V$ from a Galois object $U \in \mathrm{F\acute{E}t}_X$ (. Since V is connected, the finite étale map g is surjective i.e. an étale covering. Therefore, $p(g)\colon p(U) \twoheadrightarrow p(V)$ is surjective and thus $p(V)$ is finite.

A non-connected object $V \in \mathrm{F\acute{E}t}_X$ may be written as its connected components $V \cong \sqcup_{i\in I} V_i$. Since any finite étale map has finite fibres, the set $F_{\bar{x}}(V) \overset{2.7.4}{\cong} \sqcup_{i\in I} F_{\bar{x}}(V_i)$ is finite. Thus, $F_{\bar{x}}(V_i)$ can only be non-empty for finitely many $i \in I$. The proof of Lem. 7.3.13 shows, that $F_{\bar{x}}(W)$ is empty if and only if $p(W)$ is empty for all $W \in X_{\acute{e}t}$. Thus, $p(V) \overset{2.7.4}{\cong} \sqcup_{i\in I} p(V_i)$ is finite, since only finitely many $p(V_i)$ are non-empty and each $p(V_i)$ is finite. □

Corollary 7.3.17. *For every scheme X, any point $p\colon X_{\acute{e}t} \to \mathrm{Set}$ and any finite étale map $U \to X$, the set $p(U)$ is finite.*

Proof. Let $\bar{x}\colon \mathrm{Spec}(K) \to X$ denote the geometric point of X associated to p via Lem. 7.3.13 and let $x \in X$ be the image of $\bar{x}$. Write X as disjoint union of its connected components $X = \sqcup_{i\in I} X_i$. The proof of Lem. 7.3.13 shows, that $p(X_i) = \emptyset$ if $x \notin X_i$. Let X_j be the unique connected component, such that $x \in X_j$. Then, $* \cong p(X) \cong \sqcup_{i\in I} p(X_i) \cong p(X_j)$. This implies, that restricting p to the site $(X_j)_{\acute{e}t}$ yields a unique point

$$q\colon (X_j)_{\acute{e}t} \to \mathrm{Set}, \qquad q(U_j \to X_j) := p(U_j \to X_j \to X).$$

Composition with the inclusion $\iota_j\colon X_j \hookrightarrow X$ yields a commutative triangle

$$\begin{tikzcd}
(X_j)_{\acute{e}t} \arrow[rr, "(\iota_j)_!"] \arrow[dr, "q"'] & & X_{\acute{e}t} \arrow[dl] \\
& \mathrm{Set} &
\end{tikzcd}$$

Now, every étale map $f\colon U \to X$ is a coproduct of étale maps $f_i\colon U_i \to X_i \hookrightarrow X$. Whenever, $i \neq j$, the induced map $p(f_i)\colon p(U_i) \to p(X_i) = \emptyset$ shows that $p(U_i) = \emptyset$. Hence,

$$p(U) \cong \sqcup_{i\in I} p(U_i) \cong p(U_j) = p((\iota_j)_! U_j) = q(U_j \to X_j)$$

Since X_j is connected, the set $q(U_j)$ is finite, whenever $U_j \to X_j$ is is finite étale. □

Lemma 7.3.18. *Let $p\colon X_{\acute{e}t} \to \mathrm{Set}$ and $\bar{x}\colon \mathrm{Spec}(K) \to X$ the geometric point of X given by Lem. 7.3.13. If $F_{\bar{x}}(U) \cong *$ for an $U \in X_{\acute{e}t}$, then $p(U) \cong *$.*

Proof. Let $x \in X$ be the image of $\bar{x}$ and let $u \in U$ be the unique point lying over x. Let N be a normal closure of $\kappa(u)$ in K. Lem. 7.3.11 implies that $F_{\bar{x}}(U) \cong \mathrm{Gal}(N/\kappa(x)$. Therefore,

$$1 = |\mathrm{Gal}(N/\kappa(x)| \geq [\kappa(u) : \kappa(x)] \text{ i.e. } \kappa(u) = \kappa(x).$$

provides an open neighbourhood $V \subset X$ of x, such that $f\colon f^{-1}(V) \to V$ is an isomorphism. The open sets $f^{-1}(V)$ and $U \setminus \overline{\{u\}}$ form an open cover of U and therefore

$$* = \quad p(V) \sqcup \emptyset \ = \ p(f^{-1}(V)) \sqcup p(U \setminus \overline{\{u\}}) \longrightarrow p(U)$$

is surjective, hence $p(U) \cong *$. ☐

Question 7.3.19. Let $p\colon X_{\acute{e}t} \to \mathrm{Set}$ be a point. Let $\bar{x}\colon \mathrm{Spec}(K) \to X$ be the geometric point provided by Lem. 7.3.13 and let $x \in X$ be the image of $\bar{x}$. Let $U \to X$ be an étale morphism such that U contains only one point $u \in U$ which lies over $x \in X$. Do Lemma 7.3.18 and Lemma 7.3.17 imply $p(U)$ is a finite set? Although étale morphisms are locally quasi-finite ([Sta25, Tag 03WS]) they are not locally finite in general[63].

Theorem 7.3.20. *For any scheme X, every point $p\colon X_{\acute{e}t} \to \mathrm{Set}$ is induced by a unique geometric point of X.*

For the proof of this Theorem we assume that Question 7.3.19 can be answered positively. Under this assumption the proof strategy of [Sta25, Tag 04HU] can be applied.

Proof. Let $p\colon X_{\acute{e}t} \to \mathrm{Set}$ be a point, let $\bar{x}\colon \mathrm{Spec}(K) \to X$ be the geometric point associated to p via Lem. 7.3.13 and let $x \in X$ be image of $\bar{x}$. Pick an arbitrary finite étale morphism $U \to X$. Lemma. C.5.6 provides an étale morphism $g\colon V \to X$ such that $g^{-1}(\{x\}) = \{v\}$ for a $v \in V$ and such that there are $V_1, ..., V_n, W \in X_{\acute{e}t}$ with

$$U \times_X V \cong W \sqcup \coprod_{i=1}^{n} V_i$$

where $W \times_V \{v\} = \emptyset$ and the projection $U \times_X V \to V$ induces isomorphisms $V_i \cong V$ for each $i = 1, ..., n$. Therefore, the proof of Lem. 7.3.13 implies that $p(W) = \emptyset$. Therefore,

$$p(U) \times p(V) \cong p(U \times_X V) \cong p(W \sqcup \coprod_{i=1}^{n} V_i) \cong p(W) \sqcup \coprod_{i=1}^{n} p(V_i) \cong \coprod_{i=1}^{n} p(V)$$

$$\text{and likewise} \qquad F_{\bar{x}}(U) \times F_{\bar{x}}(V) \cong \coprod_{i=1}^{n} F_{\bar{x}}(V)$$

since p and $F_{\bar{x}}$ preserves colimits (2.7.4) and finite limits. These isomorphisms implies that $p(U)$ must and $F_{\bar{x}}$ must both be a set with n elements. Since this argument applies to all $U \in X_{\acute{e}t}$, the natural transformation $\phi\colon F_{\bar{x}} \Rightarrow p$ is an equivalence. ☐

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[63] The open immersion $\mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^1$ is not finite for any open neighbourhood of the origin 0 in the target. Otherwise, it would be surjective.

7.3.3 Local Structure of 'etale Morphisms in Analytic Geometry

To show that the étale topos of a Berkovich space admits a similar description, we recall results from [Ber93] concerning the local structure of étale morphisms. Berkovich investigates the local behavior of the étale site by defining étale germs of subsets (cf.[Ber93, §3.4]). For our purposes, it suffices to restrict our attention to the case where this subset is a point. suffices to consider only the case, where this subset is a point.

Definition 7.3.21. Let X and Y be a k-analytic spaces and $x \in X$, $y \in Y$ fixed points within these spaces. We endow the set

$$M := \{f\colon V_f \to X \mid V_f \subset Y \text{ open neighbourhood of some } y \in Y,\ f(y) = x\}$$

with an equivalence relation by writing

$$f \sim g \ :\Leftrightarrow\ \exists \text{ open neighbourhood } W_{fg} \subset V_f \cap V_g \text{ of } y \text{ such that } \ f|_{W_{fg}} \equiv g|_{W_{fg}}$$

A k-**germ** $f\colon (Y, y) \to (X, x)$ over (X, x) is an equivalence class $f \in {}^{M}\!/\!_{\sim}$.
In direct analogue to of germs in differential geometry, composition is defined by sufficient shrinking and the identity induces a neutral element with respect to composition (cf.[Dun18, §3.1]). Then, the category of k-germs $\mathrm{Germ}^k_{(X,x)}$ over (X, x) consists of germs $f\colon (Y, y) \to (X, x)$ together with a morphisms given by diagrams

$$\begin{tikzcd}
(Y,y) \arrow[rr, "f"] \arrow[dr, "g"'] & & (Z,z) \arrow[dl, "h"] \\
& (X,x) &
\end{tikzcd}$$

such that $f \simeq g \circ h$.
Then, define $\acute{\mathrm{E}}\mathrm{t}(X, x)$, $\mathrm{F}\acute{\mathrm{E}}\mathrm{t}(X, x) \subset \mathrm{Germ}^k_{(X,x)}$ as the full subcategories of germs over (X, x), which admit an étale resp. finite étale representative $f\colon (Y, y) \to (X, x)$.

Berkovich showed, that locally an étale morphism only depends on its completed residue field.

Theorem 7.3.22 ([Ber93, 3.4.1])**.** *Let X be an k-analytic space. Then for any point $x \in X$ there is an equivalence of categories*

$$\mathrm{F\acute{E}t}(X, x) \xrightarrow{\cong} \mathrm{F\acute{E}t}(\mathscr{H}(x)),\ (Y, y) \to (X, x) \ \mapsto\ \mathrm{Spec}(\mathscr{H}(y)) \to \mathrm{Spec}(\mathscr{H}(x))$$

where $\mathrm{F\acute{E}t}(\mathscr{H}(x))$ is the category of schemes, which are finite étale over $\mathrm{Spec}(\mathscr{H}(x))$.

Corollary 7.3.23. *Let $f\colon U \to X$ be a finite étale morphism and $u \in U$, $x \in X$. Suppose that*

1) *$u \in f^{-1}(\{x\})$*
2) *f induces $\mathscr{H}(x) \cong \mathscr{H}(u)$*

Then, there is an open neighbourhood $V \subset X$ of x, such that f induces an isomorphism

$$f|_{f^{-1}(V)}\colon f^{-1}(V) \to V.$$

Proof. Since u maps to x under f, the map f induces a k-germ $f\colon (U,u) \to (X,x)$. Since this germ induces an isomorphism of residue fields, it is an isomorphism of germs i.e. a local isomorphism. □

By Definition, étale morphisms of k-analytic spaces are G-locally finite, the points in their fibre can separated étale locally.

Lemma 7.3.24. *Let $f : X \to S$ be an étale morphism, $s \in S$ and $f^{-1}(s) = \{x_1, ..., x_n\}$. Then there exists an étale neighbourhood $(U,u) \to (S,s)$ and a finite disjoint union decomposition*

$$X_U = W \amalg \coprod\nolimits_{i,j} V_{i,j}$$

of schemes such that

1) *There is exactly one point $v \in V_{i,j}$ mapping to $x_i \in X$ and $V_{i,j} \times_X \{x_k\} = \emptyset$ unless $k = i$.*
2) *Any point of $(X \times_S U)_u)$ mapping to x_i is in some $V_{i,j}$.*
3) *$V_{i,j} \to U$ is an isomorphism,*
4) *The fibre W_u contains no point mapping to any x_i.*

Proof. This statement can be deduces from the proof of [Ber93, Theorem 3.4.1]. Consult also [Sta25, Tag 04HM]. □

Berkovich also proves, that the isomorphism of Thm. 7.3.22 carries over to topoi.

Theorem 7.3.25 ([Ber93, 4.2.1])**.** *For any point of an k-analytic space X, there is an equivalence*

$$Sh_{\acute{e}t}(\acute{\mathrm{E}}\mathrm{t}(X,x);\mathrm{Set}) \simeq \mathrm{Shv}_{\acute{e}t}(\mathrm{Spec}(\mathscr{H}(x));\mathrm{Set}).$$

7.3.4 Étale Topos of an analytic space

The connection between geometric points and the étale topos of a Berkovich analytic space is established in analogy to the previous discussion.

Convention 7.3.26. Throughout this subsection, let k be a non-archimedean complete valued field and X an k-analytic space.

The goal is to show that the points of $X_{an\acute{e}t}$ coincide with the geometric points defined below.

Definition 7.3.27. A geometric point is a morphism $\bar{x}\colon \mathcal{M}(K) \to X$ for an algebraically closed complete valued field K. An **étale neighbourhood** of a geometric point $\bar{x}$ is a pair $(U, \bar{u})$, where $U \to X$ is an étale morphism and $\bar{u}$ is a morphism such that the following diagram commutes

$$\begin{tikzcd}
 & U \arrow[d] \\
\mathcal{M}(K) \arrow[ur, "\bar{u}"] \arrow[r, "\bar{x}"] & X
\end{tikzcd}$$

A morphism between two étale neighbourhoods (U, u) and (V, v) of a point $\bar{x} \colon \mathcal{M}(K) \to X$ is a morphism $f \colon U \to V$ in $X_{\acute{e}t}$, which induces a commutative diagram

$$\begin{tikzcd}
 & U \arrow[d, "f"] \\
\mathcal{M}(K) \arrow[ur, "u"] \arrow[r, "v"] & V
\end{tikzcd}$$

Denote the category of étale neighbourhoods of a geometric point $\bar{x}$ by $\mathrm{Nb}_{\text{anét}}(\bar{x})$.

Observation 7.3.28. Pulling back along a geometric point $\bar{x} \colon \mathcal{M}(K) \to X$ induces points

$$F_{\bar{x}} \colon X_{an\acute{e}t} \xrightarrow{\bar{x}^*} \mathrm{Set} \qquad\qquad G_{\bar{x}} \colon \mathrm{Shv}_{an\acute{e}t}(X) \xrightarrow{\bar{x}^*} \mathrm{Shv}_{an\acute{e}t}(\mathcal{M}(K)) \overset{6.2.9}{\simeq} \mathrm{Ani}.$$

The formula 2.7.7 relates étale neighbourhoods to stalks, since for every $\mathcal{F} \in \mathrm{Shv}_{an\acute{e}t}(X)$ we get

$$G_{\bar{x}}(\mathcal{F}) \overset{2.7.7}{\simeq} \operatorname*{colim}_{(u \colon \mathcal{M}(K) \to U) \in (X_{an\acute{e}t\,\mathcal{M}(K)/})^{\mathrm{op}}} \mathcal{F}(U) \simeq \operatorname*{colim}_{(U,u) \in \mathrm{Nb}_{\text{anét}}(\bar{x})^{\mathrm{op}}} \mathcal{F}(U).$$

Lemma 7.3.29. *For any geometric point $\bar{x} \colon \mathcal{M}(K) \to X$, the category $\mathrm{Nb}_{\text{anét}}(\bar{x})$ is cofiltered.*

Proof. Let $\bar{x}$ be a geometric point. For convenience, we abbreviate $J := \mathrm{Nb}_{\text{anét}}(\bar{x})$.
First, we show that any two neighbourhoods in J admit a common refinement in J. Given $(U_i, u_i)_{i=1,2} \in J$, their pullback $U_1 \times_X U_2$ exists and is étale over X as the composition $U_1 \times_X U_2 \to U_1 \to X$ is étale by Lem. 6.2.1. By the universal property of a pullback, there is a map $u_1 \times_{\bar{x}} u_2 \colon \mathcal{M}(K) \to U_1 \times_X U_2$ which makes $U_1 \times_X U_2$ a étale neighbourhood of $\bar{x}$. Moreover, $u_1 \times_{\bar{x}} u_2$ exhibits the canonical projections to U_1 and U_2 as morphisms in J.

Given two parallel morphisms $(U, u) \underset{f_2}{\overset{f_1}{\rightrightarrows}} (V, v)$ in $\mathrm{Nb}_{\text{anét}}(\bar{x})$. To finalize the proof, we have to find an equalizing morphism in J. Let $P \in \mathrm{Ber}_k$ be the pullback defined by the pullback square

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & U \arrow[d, "(f_1 f_2)"] \\
V \arrow[r, "\Delta"] & V \times_X V
\end{tikzcd}$$

By Lem. 6.2.1.1+2, the canonical map $V \times_X V \to X$ is étale. Moreover, $V \to V \times_X V$ is étale by Lem. 6.2.1.3, since $V \xrightarrow{id_V} V$ and $V \times_X V \to V$ are étale. Applying Lem. 6.2.1.1 again, we conclude that $P \to U$ is étale and thus $P \in X_{\acute{e}t}$.

By definition of morphisms in J the following diagram commutes

$$\begin{tikzcd}
\mathcal{M}(K) \arrow[r, "u"] \arrow[d, "v"] \arrow[dr, "v \times_{\bar{x}} v"] & U \arrow[d] \arrow[dr] & \\
V \arrow[r] & V \times_X V \arrow[r] & X
\end{tikzcd}$$

Thus, there is a map $p \colon \mathcal{M}(K) \to P$, which exhibits (P, p) as an étale neighbourhood of $\bar{x}$ and the projections $(P, p) \to (U, u)$ and $(P, p) \to (V, v)$ as morphisms in J. ☐

Definition 7.3.30. Let $x\colon \mathcal{M}(K) \to X$ be a geometric point of X. Let $\mathrm{Nb}^{\mathbf{c}}_{\text{anét}}(x) \subset \mathrm{Nb}_{\text{anét}}(x)$ be the full subcategory spanned by those étale neighbourhoods $f\colon (U,u) \to (X,x)$ of x, such that $f(U)$ is a **compact** analytic domain in X. Likewise, we denote the full subcategory spanned by **affinoid** étale neighbourhoods $f\colon (\mathcal{M}(A),u) \to (X,x)$ by $\mathrm{Nb}^{\mathbf{affnd}}_{\text{anét}}(x) \subset \mathrm{Nb}_{\text{anét}}(x)$.

The following Lemma is not needed for this section, but is essential for the proof of Thm. 8.4.4.

Lemma 7.3.31. *For any geometric point x of a k-analytic space X, the inclusion* $\mathrm{Nb}^{\mathbf{c}}_{\text{anét}}(x) \subset \mathrm{Nb}_{\text{anét}}(x)$ *is* ***coinitial*** *in the sense of [Lan21, 4.4.2] i.e.* $\mathrm{Nb}^{\mathbf{c}}_{\text{anét}}(x)^{\mathrm{op}} \subset \mathrm{Nb}_{\text{anét}}(x)^{\mathrm{op}}$ *is cofinal. If X is good, then the inclusion* $\mathrm{Nb}^{\mathbf{affnd}}_{\text{anét}}(x) \subset \mathrm{Nb}_{\text{anét}}(x)$ *is* ***coinitial*** *as well.*

Before we begin with the proof, notice, that the statement above is equivalent to asserting, that the inclusions in question are coinitial/cofinal in the sense of [Lan78, p.217]. This is expatiated in the following

Observation 7.3.32. Let $f\colon C \to D$ be a functor of 1-categories. By Quillen's Theorem A [Lan21, 4.4.20] the functor $N(f)$ is cofinal if and only if $N(C)_{d/}$ is weakly contractible for every $d \in D$. Since $N(C)_{d/} \simeq N(C_{d/})$ by [Lan21, p.83], this is the case precisely if $C_{d/}$ is non-empty and connected, which is the Definition of Cofinality in [Lan78, p.217].

Proof. Let $(U,\bar{u}) \to (X,\bar{x})$ be an étale neighbourhood of $\bar{x}$. Let $u \in U$ be the point under $\bar{u}$. By Definition of k-analytic spaces there is a neighbourhood V of u which is compact analytic domain. Since U is good if X is good, we may take V to be an affinoid domain. Thus, $\mathrm{Nb}^{\mathbf{c}}_{\text{anét}/U}$ resp. $\mathrm{Nb}^{\mathbf{affnd}}_{\text{anét}}{}_{/U}$ are non-empty, since open immersions are étale. Secondly, these slice categories are connected, as pullbacks of compact analytic domains resp. affinoid spectra, which are étale over U, are compact analytic domains resp. affinoid spectra and are étale over U by Lem. 6.2.11+2. ☐

In direct analogy to Lem. 7.3.13 we conclude from the above

Lemma 7.3.33. *For any point $p\colon X_{an\acute{e}t} \to \mathrm{Set}$, there is a geometric point $\bar{x}$ of X and a natural transformation $\phi\colon F_{\bar{x}} \Rightarrow p$, where $F_{\bar{x}}\colon X_{an\acute{e}t} \to \mathrm{Set}$ is the functor given by pulling back along $\bar{x}$*

Proof. The proof strategy is the same as in Lem. 7.3.13. The composition
$q\colon \mathrm{Op}(|X|) \hookrightarrow X_{an\acute{e}t} \to \mathrm{Set}$ is a point of the site $\mathrm{Op}(|X|)$ – the site of open sets of the topological space $|X|$ underlying X. Since $|X|$ is locally Hausdorff, hence sober (7.3.7), one concludes that there is a point $x \in X$ such that q is equivalent to $p_x\colon \mathrm{Op}(X) \to \mathrm{Set}$ of Def. 7.3.2. In particular,

$$p(U) \neq \emptyset \Leftrightarrow x \in f(U) \qquad \forall \text{ étale } U \to X. \tag{$\boxed{\iota}$}$$

Let $G_{\bar{x}}, P\colon \mathrm{Shv}_{\acute{e}t}(X) \to \mathrm{Set}$ denote the essentially unique geometric morphisms induced by $F_{\bar{x}}$ and p (cf. 2.3.23). Thus, $G_{\bar{x}}$ is the stalk functor of Obs. 7.3.28. Proceeding as in the proof of Lem. 7.3.13 we conclude that

$$\mathrm{Hom}_{\mathrm{Fun}(X_{\acute{e}t},\mathrm{Set})}(F_{\bar{x}}, p) \cong \mathrm{Hom}_{\mathrm{Fun}(\mathrm{Shv}_{\acute{e}t}(X),\mathrm{Set}}(G_{\bar{x}}, P) \cong \lim_{(U,u)\in \mathrm{Nb}_{\text{anét}}(\bar{x})} p(U)$$

is a non-empty set, since $\mathrm{Nb}_{\text{anét}}(\bar{x})$ is cofiltered and $p(U) \neq \emptyset$ for all $(U,u) \in \mathrm{Nb}_{\text{anét}}(\bar{x})$. ☐

Lemma 7.3.34. *Let* $p\colon X_{an\acute{e}t} \to \mathrm{Set}$ *be a point. Let* $\bar{x}\colon \mathcal{M}(K) \to X$ *be the geometric point provided by Lem. 7.3.33 and let* $x \in X$ *be the image of* $\bar{x}$*. Let* $f\colon U \to X$ *étale morphism such that* U *contains only one point lying over* x*, then the set* $p(U)$ *is finite.*

Proof. By [Ber93, p.86], there is a morphism of presites $i\colon \acute{\mathrm{E}}\mathrm{t}(X,x) \to X_{an\acute{e}t}$. Thus,

$$q\colon \acute{\mathrm{E}}\mathrm{t}(X,x) \xrightarrow{i} X_{an\acute{e}t} \xrightarrow{p} \mathrm{Set}$$

defines a point of $\acute{\mathrm{E}}\mathrm{t}(X,x)$. Since $u \in U$ is the unique point mapping to $x \in X$ under f, there is a unique k-germ $g\colon (U,u) \to (X,x)$ represented by $f|_V\colon V \to X$ for an open neighbourhood $V \subset U$ of u. Consequently, $(U,u) \cong (V,u)$, and $i(V,u)$ is the étale morphism $f|_V\colon V \to X$. Note that q induces a unique point Q (cf. 2.7.3), which gives rise to a point

$$\mathrm{Shv}_{\acute{e}t}(\mathrm{Spec}(\mathscr{H}(x));\mathrm{Set}) \overset{7.3.25}{\simeq} \mathrm{Shv}_{an\acute{e}t}(\acute{\mathrm{E}}\mathrm{t}(X,x);\mathrm{Set}) \xrightarrow{Q} \mathrm{Set}.$$

Therefore, Cor. 7.3.17 (if U is good) or the positive answer to Question 7.3.19 implies that $p(V) = q(V) = Q(L_{an\acute{e}t}y(V))$ is finite.
As U is locally Hausdorff, we may choose V sufficiently small so that there exists a family of open subsets $\{U_i \subset U\}_{i\in I}$ where $x \notin U_i$ for all $i \in I$, and such that

$$V \sqcup \coprod_{i\in I} U_i \to U$$

is surjective. Since this forms an étale covering, the proof of Lem. 7.3.33 yields a surjection:

$$p(V) = p(V) \sqcup \coprod_{i\in I} \emptyset \overset{\boxed{\iota}}{\cong} p(V) \sqcup \coprod_{i\in I} p(U_i) \twoheadrightarrow p(U). \qquad \square$$

Lemma 7.3.35. *The assignment* $x \mapsto F_{\bar{x}}$ *yields an injection*

$$\{\textit{Geometric points of } X\} \hookrightarrow \mathrm{Fun}^*(X_{an\acute{e}t}, \mathrm{Set}).$$

Proof. Let $\bar{x}$ and $\bar{y}$ be geometric points of X, such that there is an equivalence of functors $F_{\bar{x}} \cong F_{\bar{y}}$. By Obs. 2.7.3 this is the case precisely if $G_{\bar{x}}$ and $G_{\bar{y}}$ of Obs. 7.3.28 are equivalent. Let $x, y \in X$ be the images of $\bar{x}$, $\bar{y}$ and let $i\colon \acute{\mathrm{E}}\mathrm{t}(X,x) \to X_{an\acute{e}t}$ be the associated morphism of presites (cf.[Ber93, p.86]). Then, the following compositions of geometric morphisms are equivalent

$$\mathrm{Shv}_{\acute{e}t}(\mathrm{Spec}(K);\mathrm{Set}) \overset{7.3.25}{\cong} \mathrm{Shv}_{an\acute{e}t}(\acute{\mathrm{E}}\mathrm{t}(X,x)) \to \mathrm{Shv}_{an\acute{e}t}(X) \underset{G_{\bar{y}}}{\overset{G_{\bar{x}}}{\rightrightarrows}} \mathrm{Set}.$$

Therefore, Thm. 7.3.20 implies that these compositions correspond an unique geometric point of $\mathscr{H}(x)$ i.e. to a unique field extension $K/\mathscr{H}(x)$. Thus, $\mathscr{H}(x) = \mathscr{H}(y)$. Since open immersions are étale, every open neighbourhood U of x has to contain y and vice versa. Hence, one point has to be a specialization of the other, but then $\mathscr{H}(x) = \mathscr{H}(y)$ implies that $x = y$. $\square$

Finally, we establish an analogue of Thm. 7.3.20 for Berkovich analytic spaces.

Theorem 7.3.36. *For any k-analytic space X, every point $p\colon X_{\mathit{an\acute{e}t}} \to \mathrm{Set}$ is induced by a unique geometric point of X.*

Proof. Using Lem. 7.3.24 and Lem. 7.3.34 is proved in exactly the same way as Thm. 7.3.20. ☐

As all the ∞-topoi of this section are 1-localic their points are determined by the points of the sheaves of sets. Thus, we may summarize the results of this section as follows.

Corollary 7.3.37.

- *Let X be a complex analytic space or the underlying space of a k-analytic space, then all points of $\mathrm{Shv}_{\mathit{open}}(X)$ are of the form* $\mathrm{Ani} \simeq \mathrm{Shv}_{\mathrm{open}}(*) \xrightarrow{(p_x)_*} \mathrm{Shv}_{\mathrm{open}}(X)$ *for some point $x \in X$*
- *For any scheme X all points of $\mathrm{Shv}_{\mathit{\acute{e}t}}(X)$ are of the form* $\mathrm{Ani} \simeq \mathrm{Shv}_{\mathit{\acute{e}t}}(\mathrm{Spec}(K)) \xrightarrow{\bar{x}_*} \mathrm{Shv}_{\mathit{\acute{e}t}}(X)$ *for some geometric point $\bar{x}\colon \mathrm{Spec}(K) \to X$.*
- *For any k-analytic space X all points of $\mathrm{Shv}_{\mathit{an\acute{e}t}}(X)$ are of the form* $\mathrm{Ani} \simeq \mathrm{Shv}_{\mathit{an\acute{e}t}}(\mathcal{M}(K)) \xrightarrow{\bar{x}_*} \mathrm{Shv}_{\mathit{an\acute{e}t}}(X)$ *for some geometric point $\bar{x}\colon \mathcal{M}(K) \to X$.*

Corollary 7.3.38. *Let X be a k-scheme, where k is either $\mathbb{C}$ or a non-archimedean Banach field. By Ex. 5.0.14 and Ex. 6.3.4, every point of $\mathrm{Shv}_{\mathrm{an\acute{e}t}}(X^{\mathrm{an}})$ is of the form*

$$\mathrm{Ani} \simeq \mathrm{Shv}_{\mathrm{an\acute{e}t}}(\mathrm{Spec}(K)^{\mathrm{an}}) \xrightarrow{\bar{x}_*^{\mathrm{an}}} \mathrm{Shv}_{\mathrm{an\acute{e}t}}(X^{\mathrm{an}})$$

for some geometric point $\bar{x}\colon \mathrm{Spec}(K) \to X$.

These results are particularly useful, since we can determine the stalks of sheaves at geometric points.

Corollary 7.3.39. *Let X be a k-analytic space. Let $\bar{x}\colon \mathcal{M}(K) \to X$ be a geometric point and $g^*\colon \mathrm{Shv}_{\mathit{an\acute{e}t}}(X) \to \mathrm{Shv}_{\mathit{an\acute{e}t}}(\mathcal{M}(K))$ the associated geometric morphism. Then, Obs. 7.3.28 and Lem. 7.3.31 imply*

$$g^*F \simeq \operatorname*{colim}_{(U,u)\in \mathrm{Nb}^{\mathbf{c}}_{\mathrm{an\acute{e}t}}{}^{\mathrm{op}}(\bar{x})} (F(U)) \qquad\qquad \forall\, F \in \mathrm{Shv}_{\mathit{an\acute{e}t}}(X)$$

If X is a ***good*** *k-analytic space, this formula can be simplified to*

$$g^*F \simeq \operatorname*{colim}_{(U,u)\in \mathrm{Nb}^{\mathbf{affnd}}_{\mathrm{an\acute{e}t}}{}^{\mathrm{op}}(\bar{x})} (F(U)) \qquad\qquad \forall\, F \in \mathrm{Shv}_{\mathit{an\acute{e}t}}(X)$$

8 Base Change in non-Archimedean Analytic Geometry

The study of non-abelian base change theorems dates back to the foundational work of Giraud [Gir71], who extended the classical proper base change theorem to sheaves with values in profinite groupoids over Noetherian étale schemes. The Noetherian hypothesis was subsequently eliminated in [LM14] through approximation arguments involving bands. Recently, Chough [Cho22] unified these results within the framework of sheaves with values in anima.

Theorem 8.0.1 (Proper Base Change, [Cho22, Thm.1.2])**.** *For any pullback square of quasi-compact quasi-separated schemes*

$$\begin{tikzcd}
X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
Y' \arrow[r, "g'"] & Y
\end{tikzcd}$$

where f is a proper morphism, consider the induced commutative diagram of ∞-topoi

$$\begin{tikzcd}
\mathrm{Shv}_{\acute{e}t}(X') \arrow[r, "g'_*"] \arrow[d, "f'_*"] & \mathrm{Shv}_{\acute{e}t}(X) \arrow[d, "f_*"] \\
\mathrm{Shv}_{\acute{e}t}(Y') \arrow[r, "g_*"] & \mathrm{Shv}_{\acute{e}t}(Y)
\end{tikzcd}$$

For every ***finite constructible*** *object $F \in \mathrm{Shv}_{\acute{e}t}(X)$, whose homotopy groups have torsion prime to $\mathrm{char}(\tilde{k})$, the push-pull morphism*

$$\theta \colon g^* f_* F \to g'^* f'_* F$$

is an equivalence in $\mathrm{Shv}_{\acute{e}t}(Y')$.

In [HTT, Theorem 7.3.1.18], Lurie established that the same holds true for pullback squares of locally compact Hausdorff spaces and their associated ∞-topoi.

The goal of this section is to prove an analogous result for the étale ∞-topoi of non-Archimedean analytic spaces. To do so, we first review the base change results established in [Ber93], and then develop suitable comparisons for principal G-bundles. With these classical foundations in place, we apply the reasoning of [Cho22] in conjunction with the General Comparison Theorem 4.6.2.

8.1 Abelian Base Change

Classically, proper base change results for abelian sheaves are formulated in terms of cohomology with compact support. Unlike in algebraic geometry, the direct image functor with compact support in the analytic setting of [Ber93] coincides with the standard definition topological spaces.

Definition 8.1.1 ([Ber93, p.96])**.** For every $f \colon X \to Y$ in Ber_k and $F \in \mathrm{Shv}_{\mathrm{an\acute{e}t}}(X; \mathrm{Set})$ one defines a presheaf on $Y_{an\acute{e}t}$ by setting for every $U \in Y_{an\acute{e}t}$

$$f_! F(U) := \{ s \in F(f^{-1}U) \mid \text{the morphism } f|_{\mathrm{supp}(s)} \colon \mathrm{supp}(s) \to U \text{ is compact} \}$$

Then, $f_! F$ is called the **direct image** of F **with compact support**. If f is isomorphic to the global section functor $\Gamma_{\mathrm{Shv}_{\mathrm{an\acute{e}t}}(X;\mathrm{Set})}$, then we write $H^q_c := R^q f_!$ for all q.

Lemma 8.1.2. *For every $f \colon X \to Y$ in Ber_k and $F \in \mathrm{Shv}_{\mathrm{an\acute{e}t}}(X, \mathrm{Set})$ the presheaf $f_! F$ is a sheaf.*

Observation 8.1.3. If the morphism $f \colon X \to Y$ is compact, then we have $f_! \cong f_*$

Remark 8.1.4. By [Ber93, 5.2.4], the functor $j_!$ is exact for any morphism of k-germs $j \colon (X, T) \to (X, S)$ induced by a locally closed inclusion $T \subset S$. By setting $S = X$ and $T = U$, $j_!$ is exact whenever $j \colon U \hookrightarrow X$ is an open immersion.

Definition 8.1.5 (Derived lower shriek)**.** Let $f\colon X \to Y$ be a morphism in Ber_k, which admits a **compactification**, that is f admits a factorization

$$\begin{tikzcd} X \arrow[rr, "j"] \arrow[dr, "f"'] & & \bar{X} \arrow[dl, "\bar{f}"] \\ & Y & \end{tikzcd} .$$

where j is an open immersion and $\bar{f}$ is proper. Then, set $Rf_! := R\bar{f}_* \circ j_!$.

Lemma 8.1.6. *Let X be a k-analytic space and $\bar{x}\colon \mathcal{M}(K) \to X$ a geometric point with image $x \in X$. The strict henselianization of the stalk of the usual structure sheaf $\mathcal{O}_X$ is given by*

$$\mathcal{O}^{\mathrm{sh}}_{X,x} \cong \operatorname*{colim}_{(U,u)\ \mathrm{Nb}_{\mathrm{an\acute{e}t}}(\bar{x})} \Gamma(U, \mathcal{O}_U).$$

Proof. The map $\bar{x}$ factors as

$$\mathcal{M}(K) \xrightarrow{a} \mathcal{M}(\mathscr{H}(x)) \xrightarrow{b} X.$$

In [Ber93], $\mathcal{O}_{X,x}$ is defined as the image $\mathcal{O}_X$ under $\mathrm{Shv}_{an\acute{e}t}(X) \xrightarrow{b^*} \mathrm{Shv}_{an\acute{e}t}(\mathcal{M}(\mathscr{H}(x))$. By [Ber93, 4.2.8], $\mathcal{O}^{sh}_{X,x}$ is the image of $\mathcal{O}_{X,x}$ under $\mathrm{Shv}_{an\acute{e}t}(\mathcal{M}(\mathscr{H}(x))) \xrightarrow{c^*} \mathrm{Shv}_{an\acute{e}t}(\mathcal{M}(K))$. The desired result now follows from Obs. 7.3.28. ☐

The following fundamental result from [Ber93] plays a central role in this paper.

Theorem 8.1.7 (Weak Base Change, [Ber93, 5.3.1])**.** *Let $\varphi : Y \to X$ be a Hausdorff morphism of k-analytic spaces, and let $F \in \mathrm{Shv}_{an\acute{e}t}(Y, \mathrm{Ab})$ and $x \in X$. We set*

$$Y_{\overline{x}} := Y_x \widehat{\otimes}_{\mathscr{H}(x)} \mathscr{H}(x)^a$$

and denote by F_x (resp. $F_{\overline{x}}$) the inverse image of F on Y_x (resp. $Y_{\overline{x}}$). Then for any $q \geq 0$ there is an isomorphism of $\mathrm{Gal}(\mathscr{H}(x)^a/\mathscr{H}(x))$-modules

$$(R^q\varphi_! F)_x \;\cong\; H^q_c(Y_{\overline{x}}, F_{\overline{x}}).$$

Using Thm. 8.1.7, we demonstrate that higher direct images of finite morphisms vanish. This formalizes a result utilized by Berkovich in the proof of the abelian comparison theorem [Ber93, 7.5.1], yet not explicitly proven in that work

Corollary 8.1.8 (Vanishing of finite higher direct images)**.** *Let $f\colon X \to Y$ be a finite morphism of k-analytic spaces and $F \in \mathrm{Shv}_{an\acute{e}t}(X, \mathrm{Ab})$. Then,*

1) $R^q f_* F = 0$ for all $q \geq 1$;

2) for every geometric point $\bar{y}\colon \mathcal{M}(K) \to Y$ the stalk of f_*F is given by

$$(f_*F)_{\bar{y}} \cong \bigoplus_{\bar{x}\colon \mathcal{M}(K) \to X \ s.t.\ f\circ\bar{x} = \bar{y}} F_{\bar{x}}.$$

Proof. It is convenient to establish *2)* first. Let $y \in Y$ be the image of $\bar{y}$ and abbreviate $M := \{\bar{x}\colon \mathcal{M}(K) \to X \text{ s.t. } f \circ \bar{x} = \bar{y}\}$. Consider the following pullback squares

$$\begin{tikzcd}
X_{\bar{y}} \arrow[r] \arrow[d, "g"] & X_y \arrow[r, "q"] \arrow[d] & X \arrow[d, "f"] \\
\mathcal{M}(K) \arrow[r] & \mathcal{M}(\mathscr{H}(y)) \arrow[r] & Y
\end{tikzcd}$$

Since f is finite all vertical maps are finite, which makes $X_{\bar{y}}$ a finite K-vector space i.e. $\coprod_M \mathcal{M}(K) \xrightarrow{\prod_{\bar{x}\in M}} X_{\bar{y}}$. By Cor. 2.4.11, pulling back along the $\bar{x} \in M$ induces an equivalence. Thus, the global section functor of $X_{\bar{y}}$ factors as

$$\Gamma_{\mathrm{Shv}_{an\acute{e}t}(X_{\bar{y}})} \colon \mathrm{Shv}_{an\acute{e}t}(X_{\bar{y}}) \xrightarrow{\prod_{\bar{x}\in M}} \bar{x}^* \prod_M \mathrm{Shv}_{an\acute{e}t}(\mathcal{M}(K)) \xrightarrow{\simeq} \prod_M \mathrm{Ani}$$

Here the last map is an equivalence since $\mathrm{Shv}_{an\acute{e}t}(\mathcal{M}(K)) \simeq \mathrm{Ani}$ by Ex. 6.2.9. Let $p\colon X_{\bar{y}} \to X$ be the projection. Then, Thm. 8.1.7 implies for every $F \in \mathrm{Shv}_{an\acute{e}t}(X; \mathrm{Ab})$:

$$\oplus_{\bar{x}\in M} F_{\bar{x}} \cong \Gamma(p^*F) \overset{8.1.7}{\cong} \Gamma(((f_*F))_y) \cong (f_*F)_{\bar{y}}.$$

1) follows immediately from Thm. 8.1.7, since the global section functor $\Gamma_{\mathrm{Shv}_{an\acute{e}t}(X_{\bar{y}};\mathrm{Ab})} \cong \oplus_M \Gamma_{\mathrm{Shv}_{an\acute{e}t}(\mathcal{M}(K),\mathrm{Ab})}$ is exact, because each $\Gamma_{\mathrm{Shv}_{an\acute{e}t}(\mathcal{M}(K),\mathrm{Ab})}$ is exact. □

Definition 8.1.9. A morphism of analytic spaces over k, $f\colon X' \to X$, is called *restricted* if for every pair of points $x' \in X'$ and $x = f(x') \in X$, the canonical embedding $\mathcal{H}(x) \hookrightarrow \mathcal{H}(x')$ extends to an embedding $\mathcal{H}(x)^a \hookrightarrow \mathcal{H}(x')^a$ with dense image. Typical examples include quasi-immersions and morphisms of the form $X\widehat{\otimes}\widehat{k}^a \to X$.

Corollary 8.1.10 ([Ber93, 5.3.6]). *Let $\varphi\colon Y \to X$ be a Hausdorff morphism of k-analytic spaces, and let $f\colon X' \to X$ be a restricted morphism of analytic spaces over k, which give rise to a cartesian diagram*

$$\begin{tikzcd}
Y' \arrow[r, "\varphi'"] \arrow[d, "r'"'] & X' \arrow[d, "f"] \\
Y \arrow[r, "\varphi"] & X
\end{tikzcd}$$

Then for any abelian sheaf $\mathcal{F}$ on Y and any $q \geq 0$ there is a canonical isomorphism

$$f^*(R^q\varphi_*\mathcal{F}) \simeq R^q\varphi'_*(f'^*\mathcal{F}).$$

Corollary 8.1.11. *Let H be an affinoid k-algebra, whose underlying ring is Henselian local. Let $K := \widehat{\kappa(\mathfrak{m}_H)^a}$ be the completed algebraic closure of the residue field of H. For any k-analytic space X, which is compact over $\mathcal{M}(H)$ and every $F \in \mathrm{Shv}_{an\acute{e}t}(X; \mathrm{Ab})$ Thm. 8.1.7 provides an isomorphism*

$$H^\bullet(X, F) \cong H^\bullet(X_K, F_K)$$

where $X_K := X \times_{\mathcal{M}(k)} \mathcal{M}(K)$ and F_K is the image of F under the map induced by $X_K \to X$.

Theorem 8.1.12 ([Ber93, 7.6.2]). *Let k be algebraically closed. For every k-analytic space X, any algebraically closed Banach field K extending k and every $F \in \mathrm{Shv}_{an\acute{e}t}(X; \mathrm{Ab})$ with torsion* *prime to* $\operatorname{char}(\tilde{k})$*, one has*

$$H^\bullet(X, F) \cong H^\bullet(X_K, F_K)$$

where $X_K := X \times_{\mathcal{M}(k)} \mathcal{M}(K)$ *and* F_K *is the image of* F *under the map induced by* $X_K \to X$*.*

Theorem 8.1.13 (Proper Base Change, [Ber93, 7.7.1])**.** *Let* $\varphi : Y \to X$ *be a Hausdorff morphism of* k*-analytic spaces and let* $f : X' \to X$ *be a morphism of analytic spaces over* k *which give rise to a cartesian diagram*

$$\begin{tikzcd} Y' \arrow[r, "\varphi'"] \arrow[d, "f'"'] & X' \arrow[d, "f"] \\ Y \arrow[r, "\varphi"] & X \end{tikzcd}$$

Then for any abelian sheaf $\mathcal{F}$ *on* Y *with torsion orders prime to* $\operatorname{char}(\tilde{k})$ *there is a canonical isomorphism*

$$f^*(R\varphi_!\mathcal{F}) \;\cong\; R\varphi'_!(f'^*\mathcal{F}).$$

8.2 Base Change for Principal Bundles

In order to retrieve a full account of Thm. 8.0.1 in Berkovich geometry, we need to establish analogues of Cor. 8.1.11 and Thm. 8.1.12 for principal bundles

Lemma 8.2.1. *Let* K/k *be an extension of algebraically closed, non-archimedean complete fields, so that* $i\colon k \hookrightarrow K$ *is a bounded homomorphism. For every* k*-analytic space* X*, pulling back along* i *induces an equivalence*

$$\mathrm{F\acute{E}t}(X) \to \mathrm{F\acute{E}t}(X_K),\; U \mapsto U_K$$

where we abbreviate $X_K := X \times_{\mathcal{M}(k)} \mathcal{M}(K)$ *and* $U_K := U \times_{\mathcal{M}(k)} \mathcal{M}(K)$

Proof. Recall from Def. 6.1.57, that the property of being "finite étale" is affinoid local on the source and preimages of affinoid domains in X are affinoid domains for any finite étale map. Thus, we may assume that $X = \mathcal{M}(A)$ for an affinoid k-algebra A.

Now, we claim that there is a sequence of equivalences:

$$\mathrm{F\acute{E}t}(\mathcal{M}(A)) \overset{1.}{\cong} \mathrm{F\acute{E}t}(A) \overset{2.}{\cong} \mathrm{F\acute{E}t}(A \otimes_k K) \overset{3.}{\cong} \mathrm{F\acute{E}t}(A \widehat{\otimes}_k K) \overset{4.}{\cong} \mathrm{F\acute{E}t}(\mathcal{M}(\mathcal{M}(A) \times_{\mathcal{M}(k)} \mathcal{M}(K))$$

1) Any étale Banach A-algebra B is k-affinoid, since B is a finitely presented A-algebra. Thus, the category $\mathrm{F\acute{E}t}(\mathcal{M}(A))$ of finite étale k-affinoid A-algebras is already the whole category of finite étale Banach A-algebras. Since A is Noetherian by Prop. B.0.8, all finite Banach A-algebras are determined by their underlying finite A-algebra because of Prop. B.0.10. Hence, we can drop the norm and conclude that $\mathrm{F\acute{E}t}(\mathcal{M}(A))$ is equivalent to the category $\mathrm{F\acute{E}t}(A)$ of ordinary finite étale A-algebras.

2) Cor. C.6.2 provides an equivalence of categories $\mathrm{F\acute{E}t}(A) \cong \mathrm{F\acute{E}t}(A \otimes_k K)$ of finite étale A- and $A \otimes_k K$-algebras.

3) Since A is a finite k-module, Lem. B.0.11 provides an isomorphism $A \otimes_k K \cong A \widehat{\otimes}_k K$ of finite K-algebras. Thus, they are isomorphic as A-modules as well. In particular, all finite

étale $A \otimes_k K$-algebras coincide with $A \widehat{\otimes} K$-algebras.

4) This equivalence follows by the exact same argument as the first equivalence. $\square$

Theorem 8.2.2. *Let K/k be an extension of algebraically closed fields and X a compact k-analytic space. Let $G \in \mathcal{G}\mathrm{rp}(\mathrm{Shv}_{an\acute{e}t}(\mathrm{Ber}_k))$ be a constructible discrete group sheaf with finite stalks coprime to $\mathrm{char}(\tilde{k})$. Then, pulling back along $\mathcal{M}(K) \to \mathcal{M}(k)$ induces an equivalence*

$$G\mathrm{Bun}(X) \xrightarrow{\simeq} G\mathrm{Bun}(X_K)$$

where $X_K := X \times_{\mathcal{M}(k)} \mathcal{M}(K)$.

During this proof we will often omit ”$\mathrm{Shv}_{an\acute{e}t}$” in subscripts and for example write map_{X_K} instead of $\mathrm{map}_{\mathrm{Shv}_{an\acute{e}t}(X_K)}$.

Proof. By Cor. 7.1.23, we may assume that G is a constant discrete group sheaf. As argued in Obs. 7.1.26, it suffices to show that there is an equivalence between principal G-bundles in Ber_k over X and over X_K. Now, for every principal G-bundle $P \to X$ in Ber_k there is a finite cover $\coprod_{i \in I} U_i \to X$ such that $P \times_X U_i \cong \underline{G} \times_X U_i$, where $\underline{G} \cong \coprod_{g \in G} \mathcal{M}(k)$. Since G is finite, the map $P \times_X U_i \to U_i$ is finite and therefore $P \to X$ is finite by Lem. , since I is finite. Moreover, $P \times_X U_i \cong G \times U_i \to U_i$ is étale since this is simply the projection to the second coordinate. Consider the pullback square

$$\begin{tikzcd}
P \times_X U_i \arrow[r] \arrow[d] & P \arrow[d] \\
U_i \arrow[r] & X
\end{tikzcd}$$

Using the cancellation property (Lem. 6.2.1.3) we conclude that $P \to X$ is étale because $P \times_X U_i \to P$ is étale by Lem. 6.2.1.2. Moreover, we see that composition $P \times_X U_i \to P \to X$ is étale, when we factor this morphism through the lower left corner of the diagram above. Likewise, the pullback $P_K \to X_K$ of $P \to X$ to $G\mathrm{Bun}^{\mathrm{Ber}_K}(X_K)$ is finite étale. Here $G\mathrm{Bun}^{\mathrm{Ber}_K}(X_K)$ denotes the category of principal G-bundle objects over X_K in Ber_K. Hence, there is a commutative diagram

$$\begin{tikzcd}
\mathrm{F\acute{E}t}(X) \arrow[r, "X_K \times_X -"] & \mathrm{F\acute{E}t}(X_K) \\
\cup & \cup \\
G\mathrm{Bun}^{\mathrm{Ber}_k}(X) \arrow[r, "X_K \times_X -"] & G\mathrm{Bun}^{\mathrm{Ber}_K}(X_K)
\end{tikzcd}$$

From the equivalence of Lem. 8.2.1 we conclude that $G\mathrm{Bun}^{\mathrm{Ber}_k}(X) \to G\mathrm{Bun}^{\mathrm{Ber}_K}(X_K)$ is fully faithful. Lem. 8.2.1 also implies, that for every $P_K \to X_K \in G\mathrm{Bun}^{\mathrm{Ber}_K}$ there is a $P \to X \in \mathrm{F\acute{E}t}(X)$ pulling back to $P_K \to X_K$. Since $G \cong \sqcup_{g \in G} \mathcal{M}(k)$ pulls back to $G_K \cong \sqcup_{g \in G} \mathcal{M}(K)$ and both $G \to \mathcal{M}(k)$ and $G_K \to \mathcal{M}(K)$ are finite étale, this P inherits a G-action, since $\mathrm{Hom}(P_K \times G_K, P_K) \cong \mathrm{Hom}(P \times G, P)$ by fully faithfulness. This G-action is principal, since $P_K \times G_K \cong P_K \times_{X_K} P_K$ implies $P \times G \cong P \times_X P$. Thus $P \to X$ already lies in $G\mathrm{Bun}^{\mathrm{Ber}_k}(X)$

and the lower horizontal arrow in the diagram above is essential surjective. By Obs. 7.1.26 we conclude

$$G\mathrm{Bun}(X) \xrightarrow{\simeq} G\mathrm{Bun}(X_K). \qquad \square$$

Lemma 8.2.3. *Let H be a complete henselian local k-algebra and K the completed residue field of H. For every k-analytic space X, which is* ***proper*** *over $\mathcal{M}(H)$, pulling back along $H \hookrightarrow K$ induces an equivalence*

$$\mathrm{F\acute{E}t}(X) \xrightarrow{\simeq} \mathrm{F\acute{E}t}(X_K),\, U \mapsto U_K$$

where we abbreviate $X_K := X \times_{\mathcal{M}(k)} \mathcal{M}(K)$ and $U_K := U \times_{\mathcal{M}(k)} \mathcal{M}(K)$

Proof. By virtue of Def. 6.1.57 a map into X is finite étale, if it is for every affinoid domain of X. Thus, we may assume that $X = \mathcal{M}(A)$ for an affinoid k-algebra A. As in the proof of Lem. 8.2.1, we assert that there are there is a sequence of isomorphisms

$$\mathrm{F\acute{E}t}(\mathcal{M}(A)) \overset{1.}{\cong} \mathrm{F\acute{E}t}(A) \overset{2.}{\cong} \mathrm{F\acute{E}t}(A \otimes_H K) \overset{3.}{\cong} \mathrm{F\acute{E}t}(A \widehat{\otimes}_H K) \overset{4.}{\cong} \mathrm{F\acute{E}t}(\mathcal{M}(A_K)).$$

1) This isomorphism between the categories of finite étale A-affinoid algebras and ordinary finite étale A-algebras is deduced from Prop. B.0.8 Prop. B.0.10 just like in the proof of Lem. 8.2.1.

2) A is a finite H-algebra, because of Lem. 6.1.65. Thus, Thm. C.6.1 provides an isomorphism between finite étale A-algebras and finite étale $A \otimes_H K$-algebras (without considering any norms).

3) Since A is a finite H-algebra, there is an isomorphism $A \otimes_H K \simeq A \widehat{\otimes}_H K$ as finite K-algebras, by Lem. B.0.11. Thus, they are isomorphic as A-modules as well. In particular, all finite étale $A \otimes_H K$-algebras coincide with the associated finite étale $A \widehat{\otimes}_H K$-algebras.

4) This last isomorphism is deduced in the same way as 1. $\square$

Theorem 8.2.4. *Let H be a complete henselian local k-algebra and K the completed residue field of H. For every k-analytic space X, which is* ***proper*** *over $\mathcal{M}(H)$ and any constructible discrete group sheaf $G \in \mathcal{G}\mathrm{rp}(\mathrm{Shv}_{an\acute{e}t}(X))$ pulling back along $h\colon \mathcal{M}(K) \to \mathcal{M}(H)$ induces an equivalence of ∞-categories*

$$G\mathrm{Bun}(X) \cong G_K\mathrm{Bun}(X_K)$$

where we abbreviate $X_K := X \times_{\mathcal{M}(k)} \mathcal{M}(K)$ and $G_K := h^*(G)$.

Proof. This is deduced similar to Thm. 8.2.2. $\square$

Before developing an analogous statement for Berkovich k-analytic spaces, we need to recall fundamental properties of limits in coherent topoi established in [Cho22].

8.3 Limits of coherent ∞-topoi and global sections

Generalizing [Ver+72, VI.5.1], Chough established that in the context of coherent topoi, filtered colimit preserving geometric morphism exist in great abundance.

Theorem 8.3.1 ([Cho22, Thm.3.3])**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be small ∞-categories which admit finite limits and which are equipped with finitary Grothendieck topologies. Let $f : \mathcal{C} \to \mathcal{D}$ be a continuous functor which commutes with finite limits. Let $p_* : \mathrm{Shv}(\mathcal{D}) \to \mathrm{Shv}(\mathcal{C})$ denote the induced geometric morphism of ∞-topoi. Then for each integer $n \geq 0$, the restriction of p_* to $\tau_{\leq n-1}\mathrm{Shv}(\mathcal{D})$ commutes with filtered colimits.*

For the following technical results it is convenient to take a more abstract perspective on the situation provided by Thm. 8.3.1.

Convention 8.3.2. (♣) Let $\mathcal{I}$ be a filtered ∞-category and let $p : \mathcal{I}^{op} \to \mathcal{RT}\mathrm{op}$ be a diagram of ∞-topoi $\{\mathcal{X}_i\}$. Assume that each $\mathcal{X}_i$ is coherent and that for each of the transition morphisms $p_{ij*} : \mathcal{X}_j \to \mathcal{X}_i$, the restriction of p_{ij*} to $\tau_{\leq n}\mathcal{X}_j$ the full subcategory of $\mathcal{X}_j$ spanned by the n-truncated objects commutes with filtered colimits for all integers $n \geq -1$.

For the remainder of this section, we fix a filtered category I and a diagram of ∞-topoi $p\colon I^{op} \to \mathcal{RT}\mathrm{op}$ which satisfies condition ♣. We abbreviate $\mathcal{X}_i := p(i)$ for every $i \in I$. By Prop. 2.1.5, the limit of this diagram exists. Denote it by $\mathcal{X} := \lim\limits_{I^{\mathrm{op}}}\mathcal{X}_i$.

As for 1-categories the undercategory of a filtered category is filtered and yields cofinal map. Precisely,

Lemma 8.3.3 ([HTT, 5.4.5.9])**.** *Let $\mathcal{J}$ be a filtered ∞-category and $i \in \mathcal{J}$. Then, $\mathcal{J}_{/i}$ is filtered and $t_!\colon \mathcal{J}_{/i} \to \mathcal{J}$ is cofinal i.e. restricting along $t_!$ preserves colimits (cf.[Lan21, 4.4.2]).*

Lemma 8.3.4 ([Cho22, 3.7])**.** *Consider $\mathcal{X}_i$ for a fixed $i \in I$. Let $F \in \mathcal{X}_i$ be a truncated object of $\mathcal{X}$. Then, taking global stacks provides an equivalence of anima*

$$\operatorname*{colim}_{j \in I_{i/}} \Gamma_{\mathcal{X}_j}(p_{ij}^* F) \xrightarrow{\simeq} \Gamma_{\mathcal{X}}(\pi_i^* F).$$

where $\pi\colon \mathcal{X} \underset{\pi_{i*}}{\overset{\pi_i^*}{\leftrightarrows}} \mathcal{X}_i$ is the canonical projection from the limit.

In combination with Lem. 8.3.3, this Lemma has the following immediate consequence.

Corollary 8.3.5. *Let $n \geq -1$ be an integer and $i \in I$. Suppose the projection $\pi_{i*}\colon \mathcal{X} \to \mathcal{X}$ preserves n-truncated objects. Then, cofinality of $I_{i/} \to I$ provides an equivalence*

$$\operatorname*{colim}_{j \in I} \Gamma_{\mathcal{X}_j}(\pi_j F) \simeq \Gamma_{\mathcal{X}}(F).$$

Proof. Since $\iota\colon I_{i/} \to I$ is cofinal, this map induces an equivalence $\mathrm{Ani}^{/\Gamma p} \xrightarrow{\simeq} \mathcal{RT}\mathrm{op}^{/\Gamma p\iota}$. Therefore, the vertical maps in the following commutative square are equivalences.

$$\begin{tikzcd}
\operatorname*{colim}_{j \in I} \Gamma_{\mathcal{X}_j}(\pi_j F) \arrow[r] \arrow[d, "\simeq"] & \Gamma_{\mathcal{X}}(F) \arrow[d, "\simeq"] \\
\operatorname*{colim}_{j \in I_{i/}} \Gamma_{\mathcal{X}_j} \arrow[r] & \Gamma_{\mathcal{X}_i}(F)
\end{tikzcd}$$

Now, the right vertical lower horizontal arrow in this square is an equivalence by the previous Lem. 8.3.4. The desired statement follows from the 2-out-of-3 property for equivalences[64]. ☐

8.4 Étale Compact Base Change in Berkovich Geometry

We have shown in Cor. 8.1.11 and Thm. 8.2.4 that the conditions of the Generalization method for Cohomology (Thm. 4.6.2) are satisfied. Thus, we conclude

Corollary 8.4.1. *Let H be an affinoid k-algebra, whose underlying ring is Henselian local. Let $K := \widehat{\kappa(\mathfrak{m}_H)^a}$ be the completed algebraic closure of the residue field of H. For any k-analytic space X, which is compact over $\mathcal{M}(H)$ and every **finite constructible** $F \in \mathrm{Shv}_{an\acute{e}t}(X)$ Thm. 4.6.2 provides an equivalence of anima*

$$\Gamma_{\mathrm{Shv}_{an\acute{e}t}(X)}(F) \simeq \Gamma_{\mathrm{Shv}_{an\acute{e}t}(X \times_{\mathcal{M}(H)} \mathcal{M}(K))}(h^*H).$$

Corollary 8.4.2. *Let $k \subset K$ be an extension of algebraically closed fields and let X be a **compact** k-analytic space. For any **finite constructible** object $F \in Sh_{\mathrm{an\acute{e}t}}$, whose homotopy groups have torsion prime to* char $\tilde{k}$*, Thm. 4.6.2 yields an equivalence of anima*

$$\Gamma_{\mathrm{Shv}_{\mathrm{an\acute{e}t}}(X)}(F) \to \Gamma_{\mathrm{Shv}_{\mathrm{an\acute{e}t}}(X \times_{\mathcal{M}(k)} \mathcal{M}(K))}(i^*F)$$

where $i\colon \mathcal{M}(K) \to \mathcal{M}(k)$ is the morphism induced by $k \subset K$.

Remark 8.4.3. As shown in [Ber93, 6.4.2], the assumption that F has homotopy groups prime to char $\tilde{k}$ is **necessary**. Notice, that Cor. 8.4.2 is the precise reason, why we will have to restrict to such ∞-sheaves in Thm. 8.4.4 and Thm. 9.3.3. If [Ber93, 6.4.2] would turn out false and one could omit restriction in Thm. 8.1.12, then one could also lift the restriction off Thm. 8.4.4 and Thm. 9.3.3.

One then combines these two Corollaries and § 8.3 to obtain a generalized version of the Weak Base Change Theorem 8.1.7.

Theorem 8.4.4. *Let $f\colon X \to Y$ be a **proper** morphism of **good, compact** k-analytic spaces. Let $\bar{y}\colon \mathcal{M}(K) \to Y$ be a geometric point of Y i.e. K algebraically closed. Consider the following commutative diagram in $\mathcal{RT}$op, which is obtained by taking the pullback $X_K := X \times_Y \mathcal{M}(K)$*

$$\begin{tikzcd}
\mathrm{Shv}_{an\acute{e}t}(X_K) \arrow[r, "q_*"] \arrow[d, "\Gamma_{X_K}"] & \mathrm{Shv}_{an\acute{e}t}(X) \arrow[d, "f"] \\
\mathrm{Ani} \simeq \mathrm{Shv}_{an\acute{e}t}(\mathcal{M}(K)) \arrow[r, "p_*"] & \mathrm{Shv}_{an\acute{e}t}(Y)
\end{tikzcd}$$

*Then, for every **finite constructible** $F \in \mathrm{Shv}_{an\acute{e}t}(X)$, whose homotopy groups have torsion prime to* char $\tilde{k}$*, the push-pull morphism*

$$p^* f_* F \to \Gamma_{X_K}(q^*F)$$

is an equivalence in Ani.

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[64] In this case this is simply the 2-out-of-3 property for isomorphisms of homotopy groups.

Proof. Let $y := \bar{y}(\mathcal{M}(K))$ be the unique point in the image of $\bar{y}$. Let $H := \mathcal{O}^{\mathrm{sh}}_{Y,y}$ be the strict henselianisation of $\mathcal{O}_{Y,y}$. Then, H is an affinoid k-algebra by [Ber93, 2.1.4, 4.2.8]. Let $\mathscr{H}(y)^a$ be the completed residue field of H, which is the separable algebraic closure of $\mathscr{H}(y)$. Consider the following pullback squares in Ber_k

$$\begin{tikzcd}
X_K \arrow[r] \arrow[d] & X_{\mathscr{H}(y)^a} \arrow[r] \arrow[d] & X_H \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\mathcal{M}(K) \arrow[r] & \mathcal{M}(\mathscr{H}(y)^a) \arrow[r] & \mathcal{M}(H) \arrow[r] & Y
\end{tikzcd}$$

Thus, we get the following commutative diagram in $\mathcal{RT}\mathrm{op}$

$$\begin{tikzcd}
X_K \arrow[r] \arrow[d] & X_{\mathscr{H}(y)^a} \arrow[r] \arrow[d] & X_H \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\mathcal{M}(K) \arrow[r] & \mathcal{M}(\mathscr{H}(y)^a) \arrow[r] & \mathcal{M}(H) \arrow[r] & Y
\end{tikzcd}$$

In the remainder of this proof, we denote the pullback of $F \in \mathrm{Shv}_{an\acute{e}t}(X)$ to the étale ∞-topos of X_K, X_H and $X_{\mathscr{H}(\bar{y})}$ by F_K, F_H and $F_{\mathscr{H}(y)}$, respectively. All of these sheaves are finitely constructible by . Thus, by Cor. 8.4.1 and Cor. 8.4.2, we may identify

$$\Gamma_{X_K}(F_K) \simeq \Gamma_{X_{\mathscr{H}(y)^a}}(F_{\mathscr{H}(y)^a}) \simeq \Gamma_{X_H}(F_H)$$

Therefore, it suffices to show, that

$$p^* f_* F \to \Gamma_{X_H}(F_H)$$

is an equivalence.
First of all, Lem. 7.3.28 provides the following formula for stalks of geometric points

$$p^* f_* F \simeq \operatorname*{colim}_{(U,u) \in \mathrm{Nb}^{\mathbf{affnd}^{\mathrm{op}}}_{\mathrm{an\acute{e}t}}(\bar{y})} f_* F(U) \simeq \operatorname*{colim}_{(U,u) \in \mathrm{Nb}^{\mathbf{affnd}}_{\mathrm{an\acute{e}t}}(\bar{y})^{\mathrm{op}}} F(U \times_Y X).$$

where $\mathrm{Nb}^{\mathbf{affnd}}_{\mathrm{an\acute{e}t}}(\bar{y})$ is the full subcategory of $\mathrm{Nb}_{\mathrm{an\acute{e}t}}(\bar{y})$ of affinoid étale neighbourhoods of $\bar{y}$ of Def. 7.3.30 For the second equivalence we use that $f_* F$ is the precomposition of $F \in \mathrm{Shv}(X_{an\acute{e}t})$ with $- \times_Y X \colon Y_{an\acute{e}t} \to X_{an\acute{e}t}$ by Cor. 2.3.22.

On the other hand, by Lem. 8.1.6$H = \mathcal{O}^{\mathrm{sh}}_{Y,y} = \operatorname*{colim}_{(U,u) \in \mathrm{Nb}_{\mathrm{an\acute{e}t}}(y)} \Gamma(U, \mathcal{O}_U)$ is the colimit over the filtered category $\mathrm{Nb}_{\mathrm{an\acute{e}t}}(\bar{y})$ of étale neighbourhoods of $\bar{y}$. Since the inclusion $\mathrm{Nb}^{\mathbf{affnd}}_{\mathrm{an\acute{e}t}} \subset \mathrm{Nb}_{\mathrm{an\acute{e}t}}$ is cofinal by Lem. 7.3.31, this simplifies to

$$H \cong \operatorname*{colim}_{(U,u) \in \mathrm{Nb}^{\mathbf{affnd}}_{\mathrm{an\acute{e}t}}(y)} \Gamma(U, \mathcal{O}_U) = \operatorname*{colim}_{(\mathcal{M}(A),u) \in \mathrm{Nb}^{\mathbf{affnd}}_{\mathrm{an\acute{e}t}}(y)} A.$$

Thus, the canonical map $l \colon \lim_{(U,u) \in \mathrm{Nb}^{\mathbf{affnd}}_{\mathrm{an\acute{e}t}}(\bar{y})^{\mathrm{op}}} X \times_Y U \to X_H$ is an equivalence and therefore

$$\mathrm{Shv}_{an\acute{e}t}(X_H) \simeq \lim_{(U,u) \in \mathrm{Nb}^{\mathbf{affnd}}_{\mathrm{an\acute{e}t}}(y)^{\mathrm{op}}} \mathrm{Shv}_{an\acute{e}t}(X \times_Y U).$$

Since each $U \in \mathrm{Nb}^{\mathbf{affnd}}_{\text{anét}}$ is compact, the small étale site U_{Dom} of analytic domains is finitary for each U. Thus, $\mathrm{Shv}_{an\acute{e}t}(U)$ is an coherent ∞-topos by Prop. 7.2.10. We conclude from Thm. 8.3.1, that the diagram $p\colon \mathrm{Nb}^{\mathbf{affnd}}_{\text{anét}}(\bar{y}) \to \mathcal{RT}\mathrm{op}$ obtained by setting $p(U,u) := \mathrm{Shv}_{an\acute{e}t}(X \times_Y U)$ satisfies the conditions of Cor. 8.3.5. Therefore,

$$\Gamma_{X_H}(F_H) \simeq \operatorname*{colim}_{(U,u)\in \mathrm{Nb}^{\mathbf{affnd}}_{\text{anét}}(y)^{\mathrm{op}}} \Gamma_{X\times_Y U}(F_{X\times_Y U})$$

where $F_{X\times_Y U}$ is the image of F under $l^*\colon \mathrm{Shv}_{an\acute{e}t}(X_H) \to \mathrm{Shv}_{an\acute{e}t}(X \times_Y U)$.
In summary, we have established the following of equivalences

$$p^* f_* F \simeq \operatorname*{colim}_{(U,u)\in \mathrm{Nb}^{\mathbf{affnd}}_{\text{anét}}(y)^{\mathrm{op}}} \Gamma_{X\times_Y U}(F_{X\times_Y U}) \simeq \Gamma_{X_H}(F_H) \overset{8.4.1}{\simeq} \Gamma_{X_{\mathscr{H}(y)^a}}(F_{\mathscr{H}(y)^a}) \overset{8.4.4}{\simeq} \Gamma_{X_K}(q^* F)$$

□

Remark 8.4.5. The recently proven Theorem naturally begs the question, whether the restriction to **good** k-analytic spaces is necessary or if one could use the category of étale neighbourhoods $\mathrm{Nb}^{\mathbf{c}}_{\text{anét}}(\bar{y})$ of compact analytic domains instead of $\mathrm{Nb}^{\mathbf{affnd}}_{\text{anét}}(\bar{y})$. By Lem. 7.3.31, $\mathrm{Nb}^{\mathbf{c}}_{\text{anét}}(\bar{y})\, \mathrm{Nb}_{\text{anét}}(\bar{y})$ is cofinal as well. However, we expect this approach to be doomed by the subtle defect in Berkovich geometry, that general k-analytic domains are **not** determined by global sections of their structure sheaf. Thus, although $H \cong \operatorname*{colim}_{(U,u)\in \mathrm{Nb}^{\mathbf{c}}_{\text{anét}}(\bar{y})} \Gamma_{U,\mathcal{O}(U))}$, this does **not** imply that $\mathrm{Shv}_{an\acute{e}t}(X_H) \simeq \lim_{(U,u)\in \mathrm{Nb}^{\mathbf{c}}_{\text{anét}}(\bar{y})} \mathrm{Shv}_{an\acute{e}t}(X \times_Y U)$.

Theorem 8.4.6 (Proper and Smooth Base Change)**.** *For any pullback diagram*

$$\begin{tikzcd}
X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
Y' \arrow[r, "g'"] & Y
\end{tikzcd}$$

*where f is a **proper** or **smooth** morphism of **compact** k-analytic spaces, consider the induced commutative diagram of ∞-topoi*

$$\begin{tikzcd}
\mathrm{Shv}_{an\acute{e}t}(X') \arrow[r, "g'_*"] \arrow[d, "f'_*"] & \mathrm{Shv}_{an\acute{e}t}(X) \arrow[d, "f_*"] \\
\mathrm{Shv}_{an\acute{e}t}(Y') \arrow[r, "g_*"] & \mathrm{Shv}_{an\acute{e}t}(Y)
\end{tikzcd} \tag{$\boxplus$}$$

*For every **finite constructible** object $F \in \mathrm{Shv}_{an\acute{e}t}(X)$, whose stalks are coprime to to $\mathrm{char}(\tilde{k})$, the push-pull morphism*

$$\theta\colon g^* f_* F \to g'^* f'_* F$$

is an equivalence in $\mathrm{Shv}_{an\acute{e}t}(Y')$.

Proof. Recall from Lem. 7.1.16, that finitely constructible objects are truncated coherent and that hypercompletion restricts to an equivalence $\mathcal{X}^{\mathrm{tc}} \to (\mathcal{X}^{\mathrm{hyp}})^{\mathrm{tc}}$ for any ∞-topos $\mathcal{X}$ by Cor. 2.10.7. Therefore, we may prove Thm. 8.4.6 for $\mathrm{Shv}_{an\acute{e}t}(X)^{\mathrm{hyp}}$ instead of $\mathrm{Shv}_{an\acute{e}t}(X)$. Moreover, by the universal property of hypercompleteness 2.10.4, we may replace every ∞-topos in the square $\boxplus$ with its hypercompletion. Then, $\mathrm{Shv}_{an\acute{e}t}(Y')^{\mathrm{hyp}}$ has enough points by Cor. 7.2.12 and every point of $\mathrm{Shv}_{an\acute{e}t}(Y')^{\mathrm{hyp}}$ is induced by a geometric point $p\colon \mathcal{M}(K) \to Y'$ for an algebraically closed, complete field K. By taking the pullback $X_K := X' \times_{Y'} \mathcal{M}(K)$ for any such point p, we get the following commutative diagram in $\mathcal{RT}\mathrm{op}$

$$\begin{tikzcd}
\mathrm{Shv}_{an\acute{e}t}(X'_K)^{\mathrm{hyp}} \arrow[r, "q_*"] \arrow[d, "\Gamma_{X_K}"] & \mathrm{Shv}_{an\acute{e}t}(X')^{\mathrm{hyp}} \arrow[r, "g'_*"] \arrow[d, "f'_*"] & \mathrm{Shv}_{an\acute{e}t}(X)^{\mathrm{hyp}} \arrow[d, "f_*"] \\
\mathrm{Ani} \simeq \mathrm{Shv}_{an\acute{e}t}(\mathcal{M}(K))^{\mathrm{hyp}} \arrow[r, "p_*"] & \mathrm{Shv}_{an\acute{e}t}(Y')^{\mathrm{hyp}} \arrow[r, "g_*"] & \mathrm{Shv}_{an\acute{e}t}(Y)^{\mathrm{hyp}}
\end{tikzcd}$$

Using Cor. 8.4.4 for the outer square and for the inner left square with g'^*F in place of F, we get the push-pull equivalences

$$p^*g^*f_*F \xrightarrow{\simeq} \Gamma_{X_K}(q^*g'^*F) \xleftarrow{\simeq} p^*f'_*g'^*F.$$

By commutativity of the diagram above, the induced equivalence $p^*g^*f_*F \xrightarrow{\simeq} p^*f'_*g'^*F$ is equivalent to the push-pull morphism $p^*\theta$. As this holds for any point p, we conclude that θ itself is an equivalence, since $\mathrm{Shv}_{an\acute{e}t}(Y')^{\mathrm{hyp}}$ has enough points by Cor 7.2.12. □

9 Comparison Theorems in étale Algebraic and Analytic Geometry

9.1 Comparison for Principal Bundles

The following well-known Theorem, due to Riemann, permits the extension of holomorphic functions across nowhere-dense closed subsets. This result serves as the foundation for extending finite étale coverings to finite ramified coverings, ultimately implying that $(\text{-})^{\mathrm{an}}$ induces an equivalence of finite étale coverings. The latter specialises to a comparison of principal bundles.

Theorem 9.1.1 (Riemann Extension, [Che+94, I.13.6]). *Let X be a normal complex analytic space and $Z \subset X$ be a thin (i.e. nowhere-dense) subset. Set $j\colon U := X \setminus Z \hookrightarrow X$ and let $f\colon X \setminus Z \to \mathbb{C}$ be a holomorphic function. Assume that f is locally bounded near Z or that* $\mathrm{codim}(Z, X) \geq 2$[65]*. Then, f has a unique holomorphic extension to X.*

This Theorem allows to uniquely prolong coherent sheaves from U to X.

Corollary 9.1.2 ([Ser66, §3 Proposition 4]). *In the situation of Thm. 9.1.1, in case* $\mathrm{codim}(Z, X) \geq 2$ *all coherent torsion-free sheaves are reflexive, that is $j^* \dashv j_*$ gives rise to an equivalence*

$$\{\textit{coherent, torsion-free } \mathcal{O}_U\textit{-mod}\} \underset{j_*}{\overset{j^*}{\leftrightarrows}} \{\textit{coherent, torsion-free } \mathcal{O}_X\textit{-mod}\}.$$

Bartenwerfer [Bar76] established a direct analogue of the Riemann Extension Theorem in rigid analytic geometry. By a careful reading of Bartenwerfer [Bar76, §2.10,§3.1] this Theorem can be formulated as follows in modern language[66]

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[65] Originally this condition is phrased as "thin of order 2", which is essentially the same (c.f. [Che+94, p.74]).

[66] Classical articles such as [Bar76] use "holomorphic" space for rigid analytic space, "holomorphic function" for global sections of the structure sheaf and "holomorphic set" for supports of coherent $\mathcal{O}_X$-modules (c.f. [Bar]) [Bar76, §.2.6] shows, that the set of non-singular points in a rigid space is a holomorphic set.

Theorem 9.1.3 (Bartenwerfer [Bar76, §2.10,§3.1])**.** *Let X be a normal rigid analytic space and $Z \subset X$ be nowhere-dense dense subset. Set $j\colon U := X \setminus Z \hookrightarrow X$ and let $f \in \Gamma(U, \mathcal{O}_U)$ be a global section in U. Assume, that either* $\operatorname{codim}(Z, X) \geq 2$ *or f is bounded i.e. $|f(x)| \leq 1$ for all $x \in U$. Then, there exists a unique extension of f in $\Gamma(X, \mathcal{O}_X)$. In particular, $\mathcal{O}_X \cong j_*\mathcal{O}_U$.*

Definition 9.1.4. For any open subset $U \subset X$ of $X \in \mathrm{AnSp}_{\mathbb{C}}$ or $X \in \mathrm{Ber}_k$, let $\mathrm{FEx}_{\mathbf{X},\mathbf{U}}$ denote the full subcategory of finite surjective maps $Y \to X$ such that $Y \times_X U \to U$ is a local homeomorphism or a finite étale morphism, respectively.

An analogue of Cor. 9.1.2 fails in rigid and Berkovich analytic geometry, since not all coherent sheaves are reflexive[67]. However, the fully faithfulness of extensions can still be recovered. The following approach is a combined and simplified version of [SGA1] and [Han20]. First one needs to prove, that codimension does not decrease, when passing to finite (ramified) covers.

Lemma 9.1.5. *Let $f\colon Y \to X$ be a finite surjective map of k-analytic spaces, which is étale outside a closed nowhere-dense subset $Z \subset X$ or a finite cover of complex analytic spaces. Let $V \subset X$ be a closed subset. Then,*

$$\operatorname{codim}(f^{-1}V, Y) = \operatorname{codim}(V, X).$$

Proof. For complex analytic spaces this is [Che+94, I.10.13]. Considering that the usual codimension formula is available (cf.Lem. 6.1.60) and that the dimension of an affinoid spectrum depends on its strictly k-affinoid counterpart, this follows directly from the proof of [Han20, 2.8]. □

Convention 9.1.6. For the rest of this subsection let k denote either $\mathbb{C}$ or a non-archimedean complete field.

Proposition 9.1.7. *Let X be a normal strictly k-analytic space, $Z \subset X$ a closed subset with* $\operatorname{codim}(Z, X) \geq 2$ *and $j\colon U := X \setminus Z \hookrightarrow X$ its complement. Then, the restriction functor*

$$J^*\colon \mathrm{FEx}_{X,U} \to \mathrm{F\acute{E}t}_U$$

is fully faithful.

Remark 9.1.8. In case $k = \mathbb{C}$ or $\operatorname{char}(k) = 0$ that the functor above is an equivalence even if Z is only a nowhere-dense closed subset. This was first shown by Remmert and Stein [RS53] (c.f. [SGA1, XII.5.3]) for complex analytic space. More recently, Hansen [Han20] established an analogue in rigid analytic spaces over fields of characteristic 0.

Proof. 1) J^* is faithful:
Let $f\colon Y \to X$ and $g\colon Y' \to X$ be two elements in $\mathrm{FEx}_{X,U}$. Denote $i\colon V := f^{-1}(U) \to Y$ and $i'\colon V' := g^{-1}(U) \to Y'$. Let $a, b\colon Y \to Y'$ be two parallel morphisms over X such that $a|_V = b|_V$. Then, a and b induce a morphism of $\mathcal{O}_{Y'}$-modules

$$k\colon \mathcal{O}_{Y'} \to (a_*\mathcal{O}_Y - b_*\mathcal{O}_Y).$$

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[67] Coherent torsion-free sheaves must satisfy the S_2-property to be reflexive. See [Sta25, Tag 0AVT] for the algebraic case.

Then, $\ker(k) \subset \mathcal{O}_{Y'}$ is an ideal sheaf and, by assumption, $i^*(\ker(k)) = 0$. By Lem. 9.1.5, the Riemann Extension Theorem applies to Y', V' and $g^{-1}(Z)$. Thus, $\ker(k) = 0$, since $h|_{V'} = 0$ for every $h \in \Gamma(Y', \ker(k)) \subset \Gamma(Y', \mathcal{O}_{Y'})$ implies that $h = 0$, as this is the unique extension of $h|_{V'}$.

2) J^* is full:
Let $Y, Y' \in \mathrm{FEx}_{X,U}$ be extensions of $V, V' \in \mathrm{F\acute{E}t}_U$ and let $a\colon V \to V'$ be a map over U. The task is to find a morphism b in the following diagram

$$\begin{tikzcd}
V \arrow[rr, hook, "i"] \arrow[dr, "a"] \arrow[dd, "f"'] & & Y \arrow[dr, dashed, "b"] \arrow[dd, "g"] & \\
& V' \arrow[rr, hook, "i'"] \arrow[dl, "f'"] & & Y' \arrow[dl, "g'"] \\
U \arrow[rr, hook, "j"'] & & X &
\end{tikzcd}$$

The question is local on X. Thus, in case $k \neq \mathbb{C}$, we may restrict to an open compact strictly k-analytic domain in X. Then, Y and Y' are also compact strictly k-analytic spaces. Thus, in either case, there is a version of Riemann Extension available by Theorem 9.1.1 or 9.1.3. Since $\operatorname{codim}(Z, X) \geq 2$, Lem. 9.1.5 implies that the complements of $V \subset Y$ and $V' \subset Y'$ have codimension ≥ 2 and therefore

$$i_*\mathcal{O}_V \cong \mathcal{O}_Y \qquad i'_*\mathcal{O}_{V'} \cong \mathcal{O}_{Y'}.$$

Note also, that a induces an morphism $\alpha\colon f'_*\mathcal{O}_{V'} \to f_*\mathcal{O}_V$. Applying j_* to this morphism, one obtains the morphism

$$g'_*\mathcal{O}_{Y'} \cong g'_*i'_*\mathcal{O}_{V'} \cong j_*f'_*\mathcal{O}_{V'} \to j_*f_*\mathcal{O}_V \cong g_*i_*\mathcal{O}_V \cong g_*\mathcal{O}_Y.$$

which corresponds to a morphism $b\colon Y \to Y'$ restricting to a. ☐

(-)$^{\mathrm{an}}$ sends étale morphisms to local homeomorphisms and analytifications of finite morphisms have finite fibres if $k = \mathbb{C}$. Otherwise, (-)$^{\mathrm{an}}$ also preserves étale and finite morphisms by Lem. Furthermore, recall that the order of a finite (algebraic or analytic) étale cover $f\colon Y \to X$ is the order of the finite group $\mathrm{Aut}_{\mathrm{F\acute{E}t}_X}(Y)$. The main Theorem of this section consists of showing that assignment of finite étale covers via (-)$^{\mathrm{an}}$ is an equivalence.

Theorem 9.1.9 (Comparison of Finite Coverings, [SGA1, XII.5.1], [Lüt93])**.** *Let X be k-scheme locally of finite type. Then,* (-)$^{\mathrm{an}}$ *induces an equivalence between the categories of finite étale coverings of X with order coprime to* $\mathrm{char}(k)$ *and finite open covers of X^{an} if $k = \mathbb{C}$, respectively finite analytic étale covers of X^{an} with order coprime to* $\mathrm{char}(k)$ *if $k \neq \mathbb{C}$.*

The strategy to prove this Theorem mainly consists in reducing it to the following Theorem, commonly known as GAGA , since it first appeared in Serre's famous paper [GAGA]. The non-archimedean version is due to [Köp73].

Theorem 9.1.10 ([GAGA])**.** *Let X be a projective k-variety. The category of coherent $\mathcal{O}_X$-modules is equivalent to the category of coherent $\mathcal{O}_{X^{\mathrm{an}}}$-modules.*

The literature about this Theorem is very rich. Consult for example [Nee07] for a proof over $\mathbb{C}$. There have also been advances to generalize this statement to all schemes (c.f.[Hal22]). Recently, analogues in more general settings have also been established. Porta and Yu [PY16], [Por18] proved a statement similar to Thm. 9.1.10 for higher stacks and Zavyalov [Zav24] proved an analogue for almost coherent sheaves.

The following Lemma from general point set topology will turn out useful for the proof of fully faithfulness at Thm. 9.1.9

Lemma 9.1.11. *For any topological space U and $V, W \in \mathrm{Top}_{/U}$ with V connected, there is an inclusion of sets*

$$\mathrm{Hom}_U(V, W) \subset \{\textit{ connected components of } V \times_U W\}.$$

Proof. Let $\pi_V \colon V \times_U W \to V$ denote the canonical projection. By Definition of the pullback $V \times_U W$, there is a bijection between morphisms $V \to W$ in $\mathrm{Top}_{/U}$ and sections of π_V. Clearly, the image Γ_s of a section $s\colon V \to V \times_U W$, is connected, since V is connected and s is the only section with image Γ_s. Indeed, suppose that s corresponds to $f\colon V \to W$ and there is a section t corresponding to $g\colon V \to W$ with image Γ_s. Then,

$$(y, f(y)) = s(y) = t(y') = (y', g(y))$$

implies $y = y'$, in particular $s = t$ and $f = g$. ☐

Proof of Thm. 9.1.9. **I. Fully Faithfulness.**
Let $f\colon Y \to X$ and $g\colon Z \to X$ be finite étale coverings. By considering every component individually, we may suppose that each scheme X, Y, Z. Now, $Y^{\mathrm{an}} \times_{X^{\mathrm{an}}} Z^{\mathrm{an}} \cong (Y \times_X Z)^{\mathrm{an}}$ as $(-)^{\mathrm{an}}$ is left exact. Since $(-)^{\mathrm{an}}$ induces an bijection between the connected components of of $Y \times_X Z$ and $(Y \times_X Z)^{\mathrm{an}}$ by 5.0.5, the claim follows from Lem. 9.1.11.

II. Essential Surjectivity.
Fix a finite cover $f\colon E \to X^{\mathrm{an}}$. Again, we can assume X to be connected, by considering the every connected individually.

1. Reduce to X being affine.
Suppose $X = \cup_{i\in I} U_i$ is an open affine cover with identifications $\{\phi_{ij}\colon U_i \cap U_j \xrightarrow{\cong} U_j \cap U_i\}_{i,j\in I}$ which satisfy the usual gluing conditions. Assume that for each $i \in I$ there is a finite étale $Y_i \to U_i$ such that $E|_{U_i} := f^{-1}(U_i) \cong Y_i^{\mathrm{an}}$. Set $Y_ij := Y_i \times_{U_i \cap U_j} Y_j$. The identifications ϕ_{ij} yield a gluing datum $\{\psi_{ij}\colon E|_{U_i\cap U_j} \to E|_{U_j \cap U_i}\}_{i,j}$ for E. By I. this is of the form $\{\varphi_{ij}^{\mathrm{an}} Y_{ij} \to Y_{ji}\}_{i,j}$. Let $Y := \mathrm{colim}_{i\in I}\, Y_i$ be the scheme, which arises from the gluing datum $\{\varphi_{ij}\}_{i,j}$. Then, $Y^{\mathrm{an}} \cong E$.

2. Reduce to X being a normal scheme.
Consider the normalization[68] $\eta\colon \bar{X} \to X$ of X. By construction η is integral and surjective.

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[68] Recall from [FOAG, §10.7] or [Sta25, Tag 035E], that normalization is defined for every scheme, which has locally only finitely many irreducible components, by setting

$$\tilde{X} := \coprod_{\eta \in X \text{ generic point of irreducible component}} \mathrm{Spec}(\eta) \to X.$$

Let $\check{\mathrm{C}}(\eta) := \check{\mathrm{C}}(\eta\colon \tilde{X} \to X)_\bullet \in \mathrm{Sch}^\Delta$ the check nerve. The category of descent data[69] over η $\mathrm{DescD}(\eta) := \mathrm{Fun}^{\mathrm{cart}}(\Delta^1, \mathrm{Sch}^\Delta \times_{\mathrm{Sch}^\Delta} \{\check{\mathrm{C}}(\eta)_\bullet\}$ is defined as the cartesian transformations[70] with target $\check{\mathrm{C}}(\eta)_\bullet$. Hence, pulling $\check{\mathrm{C}}(\eta^{(\mathrm{an})})$ back along each map in $X^{(\mathrm{an})}_{\acute{e}t}$ yields a commutative diagram

$$\begin{tikzcd}
{}^{\mathrm{fin}}X_{\acute{e}t} \arrow[r, "g\mapsto g^*\check{\mathrm{C}}(\eta)"] \arrow[d, "(\text{-})^{\mathrm{an}}"] & \mathrm{DescD}(\check{\mathrm{C}}(\eta)) \arrow[d, "(\text{-})^{\mathrm{an}}"] \\
{}^{\mathrm{fin}}X^{\mathrm{an}}_{an\acute{e}t} \arrow[r, "g\mapsto g^*\check{\mathrm{C}}(\eta^{\mathrm{an}})"] & \mathrm{DescD}(\check{\mathrm{C}}(\eta^{\mathrm{an}}))
\end{tikzcd} \qquad (\boxed{\mathfrak{D}})$$

where ${}^{\mathrm{fin}}X_{\acute{e}t} \subset X_{\acute{e}t}$ and ${}^{\mathrm{fin}}X^{\mathrm{an}}_{an\acute{e}t} \subset X^{\mathrm{an}}_{an\acute{e}t}$ are the full subcategories of finite étale coverings. Thus, the left vertical arrow is fully faithful. The image of $f\colon E \to X^{\mathrm{an}}$ under the lower horizontal map is the groupoid obtained by pulling the Čech nerve $\check{\mathrm{C}}(\eta^{\mathrm{an}})$ along f i.e. the diagram $f^*\check{\mathrm{C}}(\eta^{\mathrm{an}})$

$$\begin{tikzcd}
\cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & E \times_{X^{\mathrm{an}}} \tilde{X}^{\mathrm{an}} \times_{X^{\mathrm{an}}} \tilde{X}^{\mathrm{an}} \times_{X^{\mathrm{an}}} \tilde{X}^{\mathrm{an}} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & E \times_{X^{\mathrm{an}}} \tilde{X}^{\mathrm{an}} \times_{X^{\mathrm{an}}} \tilde{X}^{\mathrm{an}} \arrow[r, shift left=1] \arrow[r, shift right=1] & E \times_{X^{\mathrm{an}}} \tilde{X}^{\mathrm{an}}
\end{tikzcd}$$

By assumption, the Theorem holds for $\tilde{X}$. Thus, there is a finite étale cover $Y \to \tilde{X}$, such that $Y^{\mathrm{an}} \cong E \times_{X^{\mathrm{an}}} \tilde{X}^{\mathrm{an}}$. Hence, $f^*\check{\mathrm{C}}(\eta^{\mathrm{an}})$ is equivalent to the simplicial object $D(Y^{\mathrm{an}})_\bullet$

$$\begin{tikzcd}
\cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & Y^{\mathrm{an}} \times_{X^{\mathrm{an}}} \tilde{X}^{\mathrm{an}} \times_{X^{\mathrm{an}}} \tilde{X}^{\mathrm{an}} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & Y^{\mathrm{an}} \times_{X^{\mathrm{an}}} \tilde{X}^{\mathrm{an}} \arrow[r, shift left=1] \arrow[r, shift right=1] & Y^{\mathrm{an}}
\end{tikzcd}$$

which inherits a cartesian transformation $\alpha D(Y^{\mathrm{an}}) \to \check{\mathrm{C}}(\eta^{\mathrm{an}})$. Since $(\text{-})^{\mathrm{an}}$ commutes with finite limits, the whole diagram given by α is the image of a diagram specifying a cartesian transformation $\beta\colon D(Y) \to \check{\mathrm{C}}(\eta)$, where $\check{\mathrm{C}}(\eta)$ is the Čech nerve of η and $D(Y)$ is the simplicial object obtained from $D(Y^{\mathrm{an}}$ by omitting the '${}^{\mathrm{an}}$'. Hence, $D(Y) \in \mathrm{DescD}(\check{\mathrm{C}}(\eta))$.

By [SGA1, IX.4.1] or [Sta25, Tag 0BTP] the upper horizontal arrow in the square $\boxed{\mathfrak{D}}$ is an equivalence. Consequently, there is a finite étale map $g\colon U \to X$, such that $g^*\check{\mathrm{C}}(\eta) \cong D(Y)$. Furthermore, Lem. D.0.8 implies that the lower horizontal functor in $\boxed{\mathfrak{D}}$ is fully faithful. Thus, by commutativity of $\boxed{\mathfrak{D}}$, the fact, that

$$(g^*)^{\mathrm{an}}\check{\mathrm{C}}(\eta^{\mathrm{an}}))(g^*\check{\mathrm{C}}(\eta))^{\mathrm{an}} \cong D(Y)^{\mathrm{an}} \cong D(Y^{\mathrm{an}}) \cong f^*\check{\mathrm{C}}(\eta^{\mathrm{an}})$$

implies, that $g^{\mathrm{an}}\colon U^{\mathrm{an}} \to X^{\mathrm{an}}$ is equivalent to $f\colon E \to X^{\mathrm{an}}$

3. Reduce to X being a smooth scheme.
Let $j\colon U \hookrightarrow X$ be the open subscheme of regular points of X and let $h\colon V \to U$ be the finite étale covering, such that h^{an} coincides with the restriction of $f\colon E \to X^{\mathrm{an}}$ to f^{-1}. First. Let us argue, that there is an finite étale covering $g\colon Y \to X$ extending h. Since X is normal, the R_1 part of Serre's criterion of normality (c.f.[FOAG, Thm.26.3.2]) implies that X is regular in codimension 1. Therefore, $\mathrm{codim}(X \setminus U, X) \geq 2$. Since X is locally of finite type over k, i.e. in particular locally Noetherian, it follows that for every $z \in X \setminus U$

$$\dim \widehat{\mathcal{O}_{X,z}} = \dim \mathcal{O}_{X,z} \geq 2$$

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[69] Consult § D, for a refresher on classical descent data as presented in [SGA1, IX.§3] or [Sta25, Tag 023U].
[70] c.f. Thm. 2.4.4

Hence, $j\colon U \hookrightarrow X$ and the coherent sheaf $H := h_*\mathcal{O}_V$ satisfy the conditions of the weak algebraic Hartog's Lemma C.3.1. Consequently, j_*H is a coherent sheaf on X and thus corresponds to a finite étale covering $g\colon Y \to X$. Now, g^{an} and $f\colon E \to X^{\mathrm{an}}$ both restrict to $h^{\mathrm{an}}\colon V^{\mathrm{an}} \to U^{\mathrm{an}}$ and therefore, $f \cong g^{\mathrm{an}}$ by Prop. 9.1.7.

4. Main Proof for the Case[71]*:* $\operatorname{char} k = 0$
Resulting from the previous reductions X can assumed to be an affine smooth normal k-scheme of finite type. Therefore, X is isomorphic to its normalization, hence is a disjoint union of normal, smooth affine k-varieties[72] (c.f.[Sta25, Tag 0357]). Because it suffices to prove this Theorem for every summand, we may take X to be a normal smooth affine k-variety. As such, X admits a dense open immersion $X \mathrel{-\!\!\circ\!\!\rightarrow} P$ into a smooth projective k-scheme P by an application of Hironaka's Resolution of Singularities on its embedding into projective space (c.f. Cor. C.2.3). It suffices to show that the given cover $f\colon E \to X^{\mathrm{an}}$ prolongs to a finite cover of $g\colon E' \to P^{\mathrm{an}}$, which is automatically unique by Prop. 9.1.7. Once this is achieve, the GAGA Theorem 9.1.10 shows that there is a finite étale cover $h\colon Y \to P$, such that $g \cong h^{\mathrm{an}}$, hence $(h|_X)^{\mathrm{an}} \cong f$. By Prop. 9.1.7 it also suffices to pick a neighbourhood U in P^{an} for each point the nowhere-dense set $P^{\mathrm{an}} \setminus X^{\mathrm{an}}$ and find an extension of $V \cap X^{\mathrm{an}}$ to P^{an}. Over $\mathbb{C}$ this is a simple task.

4.1. Case: $k = \mathbb{C}$.
Since P is smooth, P^{an} does not admit singular points, thus is a complex manifold (c.f. [GR65, A.V.9]). Therefore, every $x \in P^{\mathrm{an}} \setminus X^{\mathrm{an}}$ admits sufficiently small a open neighbourhood $U \in P^{\mathrm{an}}$, which is isomorphic to a open ball B^n in $\mathbb{C}^n$ for some n with coordinates $z_1, ..., z_n$ such that $U \setminus U \cap X^{\mathrm{an}} = \{x \in U \mid 0 = x_1, ..., x_p\} = V(z_1, ..., z_p)$ is the zero-set of the coordinates $z_1, ..., z_p$ for $1 \leq p \leq n$. Write $B^n = B^p \times B^{n-p}$ for some open balls $B^p \subset \mathbb{C}^p$ and $B^{n-p} \subset \mathbb{C}^{n-p}$. Then, $T := U \cap X^{\mathrm{an}}$ is isomorphic to the punctured open ball $B^p \setminus \{0\}$. Hence $\pi_1 T = \mathbb{Z}^p$, since T is the p-dimensional torus. The finite covering $q := f|_{f^{-1}T}\colon f^{-1}(T) \to T$ is Galois, since $q_*(\pi_1(f^{-1}T)) \subset \pi_1 T$ is clearly a normal subgroup. The finite group of Deck transformations of q is of the form $\operatorname{Deck}(q) \cong \mathbb{Z}/n_1 \times ... \times \mathbb{Z}/n_p$ for $n_1, ..., n_p \geq 1$. In particular, q is given by

$$q\colon f^{-1}(T) \to f^{-1}T/\operatorname{Deck}(p) \cong T, \qquad Z_i \mapsto Z_i^{n_i} = z_i.$$

Hence, $f^{-1}(T)$ is the quotient $T[Z_1, ..., Z_p]/(Z^{n_i} - z_i) := T \times \mathbb{C}^p/(z_i \sim Z_i^{n_i})$, where z_i are the coordinates of T and Z_i the coordinates of $\mathbb{C}^p$. This covering prolongs to the ramified cover

$$E' := B^p[Z_1, ..., Z_p]/(Z_i^{n_i} - z_i) \to E'/\operatorname{Deck}(q) \cong B^p, \qquad Z_i \mapsto Z_i^{n_i} = z_i.$$

Thus, $E' \times B^{n-p} \to B^n \cong U$ is a finite cover extending E.

4.2 Case $k \neq \mathbb{C}$.
This case is more subtle and relies on the following Lemmas, which are stated without proof.

Lemma 9.1.12 ([Lüt93, Lem.3.3])**.** *Let* $\mathbb{B}^r := k\{T_1, ..., T_n\}$ *be the r-dimensional unit ball. Let*

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[71] Possibly one could include all fields k, which allow a Resolution of Singularities here (c.f. Def C.1.5). Unfortunately, the Theorems of [Lüt93], which are used in case $k \neq \mathbb{C}$, are only formulated for $\operatorname{char} k = 0$. However, the proof of these Theorems seems to work conceptually, so that it is expected that they hold regardless of the characteristic.

[72] i.e. separated integral schemes of finite type over k.

S be a smooth strictly k-affinoid space, and let $r \geq 1$ be any integer. If Y_0 is a cover of $S \times (\mathbb{B}^r \setminus V(T_1, \dots, T_r))$ which is étale over $S \times (\mathbb{B}^r \setminus V(X_1 \cdots X_r))$, then Y_0 extends to a cover $\tilde{Y}$ of $S \times \mathbb{B}^r$.

Lemma 9.1.13 ([Kie67, 1.18], [Mit09, 2.12], [Han20, 2.13]). *If $D \subset X$ is a* ***strict normal crossings divisor*** *in a smooth strictly k-analytic space, then for any point x in X contained in exactly r irreducible components $D_1, \dots, D_r$ of D, we can find some small open strictly affinoid $U \subset X$ containing x together with a smooth strictly affinoid S and an isomorphism $U \simeq S \times \mathbf{B}^r$, under which the individual components $D_i \cap U$ containing x identify with the zero loci of the coordinate functions $X_i \in \mathcal{O}(\mathbf{B}^r)$.*

As sketched in [Han20, 2.10][73] , the covering $f \colon E \to X^{\mathrm{an}}$ can now be extended over the divisor with strict normal crossings $D = P^{\mathrm{an}} \setminus X^{\mathrm{an}}$ in the following way. For any $x \in X^{\mathrm{an}}$ denote the number of irreducible components of D passing through x by $r(x) \geq 0$. Set
$\iota(D) := \max_{x \in X} r(x)$ and continue with an induction on $\iota(D)$. Fix a point $x \in X^{\mathrm{an}}$. If $\iota(D) = 1$, then $D \subset P^{\mathrm{an}}$ consists of only one irreducible component. Thus, Lem. 9.1.13 provides an smooth strictly affinoid S such that $S \times \mathbb{B}^1$ forms an open cover for x and Lem. 9.1.12 produces a local extension of the covering.
In case $\iota(D) \geq 2$, it suffices to consider the case that $r := r(x) = \iota(D)$. Suppose $D_1, .., D_r$ are the irreducible components containing x and let $U = S \times \mathbb{B}^r$ be the strictly affinoid neighbourhood of x provided by Lem. 9.1.12. By the induction hypothesis, there are finite étale extensions $f_i \colon E_i \to U \setminus D_i$ of $f|_{U \cap X^{\mathrm{an}}}$ for each $1 \leq i \leq r$, since $r(y) < \iota(D)$ for all $y \in U \setminus D_i$. As $\{U \setminus D_i \to U\}_i$ forms an open covering, these f_i can be glued to a finite étale cover

$$g \colon E' \to U \setminus \cap_{i=1}^r D_i = S \times \mathbb{B}^r \setminus V(z_1, ..., z_r).$$

Finally, Lem. 9.1.12 provides an extension of g to whole of U.

5. *Main Proof for the Case:* $\mathrm{char}(k) \neq 0$.
By reductions 1)-3), X can be assumed to be an affine smooth normal scheme. Decomposing X into its connected components – just as done in 4) – reduces us to the case that X is an affine smooth k-variety. As such, X admits a dense open immersion $X \to Q$ into a projective k-scheme Q. Since X is normal, there exists a unique dense open immersion $j \colon X \to \tilde{Q}$ into the normalization $n \colon \tilde{Q} \to Q$ of Q. Since n is a finite surjective morphism, $\tilde{Q}$ is also a projective k-scheme[74]. Hence, it suffices to show, that the give finite étale cover $f \colon E \to X^{\mathrm{an}}$ extends to a finite (ramified) cover on $\tilde{Q}^{\mathrm{an}}$, as by GAGA (Thm. 9.1.10) any such extension coincides with an algebraic cover. Denote by $R := \tilde{Q}^{sing} \subset \tilde{Q}$ the closed nowhere-dense subset of singular points. Consider the dense open $X \hookrightarrow \tilde{Q} \setminus R$. Then, the argument of 4) applies, since $\tilde{Q} \setminus R$ is smooth and we obtain a finite morphism $g \colon Y \to \tilde{Q} \setminus R$ such that g^{an} extends the given finite étale f. Note that, $\mathrm{codim}(R, \tilde{Q}) \geq 2$ since $\tilde{Q}$ is normal. Lütkebohmert [Lüt93, §4] then claims that the newly obtained g extends to a finite covering $\tilde{Q} \setminus S$, where $S \subset \tilde{Q}$ is a closed

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[73] However, the outline in [Han20] contains a minor inaccuracy, as it seems that global covers are constructed in the induction step. This step fails, since it assumes that taking out an component D_i reduces the number $\iota(D)$ globally, which is not the case.

[74] This is [GW20, Cor.13.77]. This fact can also be seen by [Har77, III.Ex.5.7] and Serre's Theorem A, which implies that ample line bundles are very ample over projective schemes over Noetherian rings.

subset with $\operatorname{codim}(S, \tilde{Q}) \geq 3$, by blowing up locally once. More precisely, by Thm. 9.1.7 it suffices to establish an extension of g^{an} locally on R. Let η be the generic point of an irreducible component of R and set $P := \operatorname{Spec}(\mathcal{O}_{\tilde{Q},\eta})$. Hence, $\dim P = 2$ and P is a normal excellent scheme of finite type over k. Lütkebohmert [Lüt93, §4] then asserts, that by [Lip78] and/or [Abh66, 0.2] there exists a blow-up $\pi\colon P' \to P$ along an ideal of codimension 2, such that P' is non-singular and the inverse image $D := \pi^{-1}(R_\eta) \subset P'$ of the fiber $R_\eta \subset P$ is a divisor with normal crossings. Although, this seems reasonable, the references [Lip78] and [Abh66] do not contain this statement involving normal crossing divisors. The existence of such normal crossing divisor D is necessary however, as one could then continue as in 4) and use an induction on the irreducible components passing through a point. The induction start is solved in [Lüt93, 2.11] or equivalently [Ber93, 6.3.5]. The induction step relies on [Lüt93, 4.2]. Consequently, Lütkebohmert's proof relies on the existence of the normal crossing divisor D, and it will be a future task to determine the correctness of this assertions. ☐

Corollary 9.1.14. *For every qcqs scheme X, which is locally of finite type over k, and any truncated coherent discrete group sheaf $G \in \mathcal{G}\mathrm{rp}(\mathrm{Shv}_{\acute{e}t}(X))$ with stalks coprime to* $\operatorname{char}(k)$*, the functor* $(\text{-})^{\mathrm{an}}$ *induces an equivalence*

$$G\mathrm{Bun}_{\mathrm{Shv}_{\acute{e}t}(X)} \simeq G^{\mathrm{an}}\mathrm{Bun}_{\mathrm{Shv}_{an\acute{e}t}(X^{\mathrm{an}})}$$

of categories of principle G-∞-bundles.

Proof. Recall from Prop. 7.1.18 that all truncated coherent objects in $\mathrm{Shv}_{\acute{e}t}(X)$ are finitely constructible. After identifying each category of principle ∞-bundles with global sections of via Thm. 4.2.8, one applies Cor. 7.1.23 and deduces, that it suffices to prove the assertion whenever G is constant. Thus, G may be assumed to lie in the image of $\mathcal{G}\mathrm{rp}(\mathrm{Ani}^{\pi} \cap \mathrm{Ani}^{\operatorname{char}(k)\text{-coprime}})$ under $\underline{\Delta}$, which is precisely the ordinary category of finite groups with order coprime to $\operatorname{char}(k)$. In sum, it suffices to show

$$\underline{G}\mathrm{Bun}_{\mathrm{Shv}_{\acute{e}t}(X)} \simeq \underline{G}\mathrm{Bun}_{\mathrm{Shv}_{an\acute{e}t}(X^{\mathrm{an}})}$$

for any finite group discrete G with order coprime to $\operatorname{char}(k)$. By Prop. 7.1.26, every principal $\underline{G}$-∞-bundle in each topos is represented by an ordinary principal $\underline{G}$-bundle. Hence, $(\text{-})^{\mathrm{an}}$ induces an functor

$$\underline{G}\mathrm{Bun}_X \xrightarrow{(\text{-})^{\mathrm{an}}} \underline{G}\mathrm{Bun}_{X^{\mathrm{an}}}$$

of principal $\underline{G}$-bundles over X in $\mathrm{Sch}^{\mathrm{lft}}_{/k}$ and principal $\underline{G}$-bundles over X^{an} in Ber^{good}_k resp. $\mathrm{AnSp}_{\mathbb{C}}$ and it suffices to show that this is an equivalence. However, every principal $\underline{G}$-bundle $P \to X$ is in particular a finite étale morphism, since it is étale locally a trivial $\underline{G}$-bundle. For the same reason, every $\underline{G}$-bundle $Q \to X^{\mathrm{an}}$ is a finite étale morphism in case $k \neq \mathbb{C}$. Otherwise, it is a finite covering since G is a finite discrete group.

1. *Fully faithfulness.*
By Lem. 4.2.7 every map between principal ∞-bundles is an equivalence. Since the Yoneda-embedding sends principal bundles to principal ∞-bundles (c.f. Ex. 4.2.5), it suffices to observe,

that by Thm. 9.1.1

$$P \cong Q \text{ if and only if } P^{\mathrm{an}} \cong Q^{\mathrm{an}} \qquad \forall P, Q \in \underline{G}\mathrm{Bun}_X$$

since P, Q are finite étale over X.

2. *Essential surjectivity.*
Let $p\colon P \to X^{\mathrm{an}}$ be a principal $\underline{G}$-bundle, i.e. there is an action $a\colon P \times \underline{G} \to P$, such that

$$P \times \underline{G} \xrightarrow{a \times pr_1} P \times P, \qquad\qquad \mathrm{coeq}(P \times \underline{G} \underset{pr_P}{\overset{a}{\rightrightarrows}} P \times P) \to X^{\mathrm{an}}$$

are isomorphisms (c.f. Ex. 4.2.5). By Thm. 9.1.1 there is a finite étale map $q\colon Q \to X$, such that $q^{\mathrm{an}} \cong p$. Since $\underline{G}$ is a finite discrete group object, $a, pr_1\colon P \times \underline{G} \to P$ are both finite étale resp. finite coverings. Thus, there is a unique action $b\colon Q \times \underline{G} \to Q$, such that $a \cong b^{\mathrm{an}}$ and by Thm. 9.1.9

$$Q \times \underline{G} \xrightarrow{a \times pr_1} Q \times Q, \qquad\qquad \mathrm{coeq}(Q \times \underline{G} \underset{pr_Q}{\overset{a}{\rightrightarrows}} Q \times Q) \to X$$

are isomorphisms in $\mathrm{Sch}^{\mathrm{lft}}_{/k}$ i.e. b is a principal $\underline{G}$-action with quotient X. By Prop.4.2.6, local triviality of $Q \to X$ is not required in order to obtain a equivalence of principal ∞-bundles. $\square$

9.2 Comparison of Étale Abelian Cohomology

Convention 9.2.1. Throughout this section, let k be either a non-archimedean, complete field or $k = \mathbb{C}$. Denote by $\widetilde{k}$ the residue field of k.

Theorem 9.2.2 (Proper Comparison, [Ber93, 7.1.1])**.** *Let $f\colon X \to S$ be a proper morphism in $\mathrm{Sch}^{\mathrm{fin}}_{/k}$. Let $F \in \mathrm{Shv}_{\acute{e}t}(X, \mathrm{Ab})$ be a sheaf with stalks in finite abelian groups that have order coprime to $\mathrm{char}(k)$. Then, there is an equivalence in $\mathrm{D}^+(S^{\mathrm{an}})$*

$$(Rf_*F)^{\mathrm{an}} \simeq (Rf_*^{\mathrm{an}})F^{\mathrm{an}}$$

Corollary 9.2.3 (Smooth Comparison, [Ber93, 7.5.2])**.** *Let X be a smooth k-scheme locally of finite type. Let $F \in Sh_{\acute{e}t}(X, \mathrm{Ab})$ be constructible abelian sheaf with finite stalks, whose order is coprime to $\mathrm{char}\,\widetilde{k}$. Then,*

$$H^\bullet(X, F) \cong H^\bullet(X^{\mathrm{an}}, F^{\mathrm{an}}).$$

The goal of this section is to reprove the comparison Theorem [AGVb, XVI.Thm.4.1] of the étale cohomology of a wide class of discrete torsion abelian sheaves with the étale cohomology of their their analytifications only for the case of complex numbers, but also for non-archimedean fields. The latter has been achieved in [Ber95, Thm.3.1], where Berkovich generalized his earlier Comparison Theorem [Ber93, Thm.7.5.3] for fields of mixed characteristic. Precisely, we will prove

Theorem 9.2.4. *Let $X \to S$ be a morphism of schemes locally of finite type over k, where k is either a non-archimedean field or $\mathbb{C}$. Let $F \in \mathrm{Shv}_{\acute{e}t}(X, \mathrm{Ab})$ be a constructible sheaf with stalks* *in finite abelian groups that have order coprime to* $\mathrm{char}(k)$*. Then, there is an equivalence in* $\mathrm{D}^+(S^{\mathrm{an}})$

$$(Rf_*F)^{\mathrm{an}} \simeq (Rf_*^{\mathrm{an}})F^{\mathrm{an}}$$

Note. Before we start with the proof, we want to remark that derived functors commutes with composition, because of the Grothendieck spectral sequence [Wei94, Cor.10.8.3]. This fact is constantly used in the proof.

Remark 9.2.5. Let $j\colon U \hookrightarrow X$ be an open subscheme and $Z := X \setminus U \overset{i}{\hookrightarrow} X$ the associated complement. Fix a $G \in \mathrm{Shv}_{ét}(X, \mathrm{Ab})$. As in [Har77, II.Ex.1.20] define the subsheaf $\mathcal{H}_Z(G) \subset G$ of sections with support in Z by setting

$$\mathcal{H}_Z(G)(U) := \{s \in G(U) \mid \mathrm{supp}(s) \subset U \times_X Z\}$$

for every $U \in X_{ét}$. Hence, $\mathcal{H}_Z(G)$ the biggest subsheaf of G with support in Z and thus lies in the essential image of the fully faithful functor $i_*\colon \mathrm{Shv}_{ét}(Z; \mathrm{Ab}) \to \mathrm{Shv}_{ét}(X; \mathrm{Ab})$ (see [Sta25, Tag 04CA]). Usually $\mathcal{H}_Z(G)$ is regarded as sheaf on $Z_{ét}$. The assignment

$$\mathcal{H}_Z\colon \mathrm{Shv}_{ét}(X, \mathrm{Ab}) \to \mathrm{Shv}_{ét}(Z, \mathrm{Ab}),\ G \mapsto \mathcal{H}_Z(G)$$

provides a left exact functor by [Sta25, 09XP], since $\mathcal{H}_Z$ is right adjoint to i_*. Similar to [Har77, II.Ex.1.20], there is a short exact sequence for any injective $I \in \mathcal{O}_X$-Mod by [Sta25, 09XR].

$$0 \longrightarrow i_*\mathcal{H}_Z(I) \longrightarrow I \longrightarrow j_*j^*I \longrightarrow 0$$

As the functors i_* and j^* are exact, this yields the following distinguished triangle

$$i_*R\mathcal{H}_Z(G) \longrightarrow G[0] \longrightarrow Rj_*j^*G \longrightarrow i_*R\mathcal{H}_Z(G)[1] \tag{$\triangle$}$$

Note. The Proof of Thm. 9.2.4 combines the ideas of Berkovich Comparison Theorems [Ber93, 7.5.1] and [Ber95, 3.1] to a slightly more compact outline for example by using arguments in derived categories instead of spectral sequences.

Proof of Thm. 9.2.4. First, we invoke a series of reduction steps.

1. Reduce to F constant.

By Cor. 7.1.23 it suffices to show that ∞-cohomology coincides on π-finite constant sheaves, in order to retrieve a comparison for finitely constructible sheaves. By Ex. 4.1.12, this result specialises to classical abelian cohomology.

2. Reduce to $f\colon X \to S$ being a dense open immersion

f is locally of finite type by cancellation property of morphisms locally of finite type ([Sta25, Tag 01T8]). As the question is Zariski-local on S and X, we may assume that X is affine and f is of finite type. In this case, the cancellation property of separated morphisms ([Sta25, Tag 01KV]) implies that f is separated as well. Using Nagata compactification (Thm. C.2.1), there is a commutative diagram

$$\begin{tikzcd} X \arrow[rr, hook, "j"] \arrow[dr, "f"'] & & \overline{X} \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

where j is dense open immersion and g is proper. Thus, by Thm. 9.2.2 there is an isomorphism $(Rg_*(G))^{\mathrm{an}} \cong Rg_*^{\mathrm{an}}(G^{\mathrm{an}})$ for every $G \in \mathrm{D}^+_{\mathrm{tors}}(\overline{X})$. In view of the asserted reduction, suppose Thm. 9.2.4 holds for j. Since Rj_*F is constructible by, this provides us with a chain of equivalences

$$(Rf_*(F))^{\mathrm{an}} \cong (Rg_* \circ Rj_*(F))^{\mathrm{an}} \cong Rg_*^{\mathrm{an}}(Rj_*(F))^{\mathrm{an}} \cong Rg_*^{\mathrm{an}} \circ Rj_*^{\mathrm{an}}(F^{\mathrm{an}}) \cong Rf_*^{\mathrm{an}}(F^{\mathrm{an}}).$$

In consequence, it suffices to prove Thm. 9.2.4 for all dense open immersions.

All further reduction steps require an induction on the dimension of X. It is therefore helpful to invoke the reduction once "globally", before we carry on with the proof. Thus, we prove

Induction start: d:=dim(X)=0
For each $x \in X$, the local ring $\mathcal{O}_{X,x} \cong \mathcal{O}_{S,j(x)}$ has an unique prime ideal and every $y \in S$ specialises to an $x \in X$. Thus $j(X)$ is stable under specialisation in S. In particular j is quasi-compact because S is. Therefore, [Har77, II.4.5] implies, that $j(X)$ is closed in S, hence $X \cong S$.

Induction Step: d-1 ⇒ d
Let $\dim(X) = d$ for some $d \geq 1$ and assume Thm. 9.2.4 applies for all morphisms $g\colon Y \to T$, whenever $\dim(Y) \leq d-1$.

If the field k allows a Resolution of Singularities[75] – i.e. in particular if $\mathrm{char}(k) = 0$ – this Theorem can be used for a substantially easier proof.

3. Case: *k allows a Resolution of Singularities*
3.1 Reduce to S being a smooth scheme
Using Hironaka's Resolution of Singularities C.1.3, there is a smooth scheme $\tilde{S}$ and a projective birational morphism $\varphi\colon \tilde{S} \to S$. Since Resolution of Singularities C.1.3 is functorial wrt. smooth morphisms, pulling back φ along the open immersion f provides a resolution of singularities $\psi\colon \tilde{X} \to X$. Let $j\colon U \hookrightarrow X$ be the open subscheme of regular points and $\tilde{U} := \psi^{-1}(U)$. The situation is summarized with the commutative diagram

$$\begin{tikzcd} \tilde{U} \arrow[r, hook, "\tilde{j}"] \arrow[d, "\cong"] & \tilde{X} \arrow[r, "\tilde{f}"] \arrow[d, "\psi"] & \tilde{S} \arrow[d, "\varphi"] \\ U \arrow[r, hook, "j"] & X \arrow[r, "f"] & S \end{tikzcd}$$

Denote the complements by $Z := X \setminus U$ and $\tilde{Z} := \tilde{X} \setminus \tilde{U}$ and view them as closed subschemes equipped with the reduced structure i.e. we have closed immersions $i\colon Z \to X$ and $\tilde{i}\colon \tilde{Z} \to \tilde{X}$. For simplicity, we abbreviate $F_{\tilde{X}} := \psi^*F$, $F_{\tilde{Z}} := \tilde{i}^*\psi^*F$ and $F_{\tilde{U}} := \tilde{j}^*\psi^*F$.

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[75] c.f Def C.1.5

By Prop. 2.5.6 there is a short exact sequence

$$0 \to \tilde{j}_! F_{\tilde{U}} \to F_{\tilde{X}} \to \tilde{i}_* F_{\tilde{Z}} \to 0.$$

Applying Rf_* and (-)$^{\mathrm{an}}$ yields a morphism of distinguished triangles

$$\begin{tikzcd}
(R\tilde{f}_*(\tilde{j}_! F_{\tilde{U}}))^{\mathrm{an}} \arrow[r] \arrow[d] & (R\tilde{f}_* F_{\tilde{X}})^{\mathrm{an}} \arrow[r] \arrow[d] & (R\tilde{f}_*(\tilde{i}_* F_{\tilde{Z}}))^{\mathrm{an}} \arrow[r] \arrow[d] & (R\tilde{f}_*(\tilde{j}_! F_{\tilde{U}}))^{\mathrm{an}}[1] \arrow[d] \\
R\tilde{f}^{\mathrm{an}}_*(\tilde{j}_! F_{\tilde{U}})^{\mathrm{an}} \arrow[r] & R\tilde{f}^{\mathrm{an}}_* F^{\mathrm{an}}_{\tilde{X}} \arrow[r] & R\tilde{f}^{\mathrm{an}}_*(\tilde{i}_* F)^{\mathrm{an}} \arrow[r] & R\tilde{f}^{\mathrm{an}}_*(\tilde{j}_! F_{\tilde{U}})^{\mathrm{an}}[1]
\end{tikzcd}$$

To carry out the asserted reduction, suppose Thm. 9.2.4 holds whenever the base scheme smooth. This assumption implies

$$(R\tilde{f}_*(F_{\tilde{X}})^{\mathrm{an}} \cong R\tilde{f}^{\mathrm{an}}_*(F^{\mathrm{an}}_{\tilde{X}}) \text{ and } (R\tilde{f}_*(i_* F_{\tilde{Z}})^{\mathrm{an}} \cong R\tilde{f}^{\mathrm{an}}_*(i_* F_{\tilde{Z}})^{\mathrm{an}}$$

since $\tilde{i}_* F_{\tilde{Z}}$ is constructible as $\tilde{i}$ is finite and $F_{\tilde{Z}}$ constant. From the morphism of distinguished triangles, one concludes

$$(R\tilde{f}_*(\tilde{j}_! F_{\tilde{U}}))^{\mathrm{an}} \cong R\tilde{f}^{\mathrm{an}}_*(\tilde{j}_! F_{\tilde{U}}))^{\mathrm{an}}.$$

However, since ψ induces an isomorphism $\tilde{U} \cong U$ and φ is proper, we get a chain of isomorphisms in $\mathrm{D}(S^{\mathrm{an}})$

$$\begin{aligned}
(Rf_*\,(j_! F|_U))^{\mathrm{an}} &\cong (R(\varphi\tilde{f})^{\mathrm{an}}\,(\tilde{j}_! F_{\tilde{U}}))^{\mathrm{an}} \cong (R\varphi_* R\tilde{f}_*\,(\tilde{j}_! F_{\tilde{U}}))^{\mathrm{an}} \\
&\cong R\varphi^{\mathrm{an}}_* R\tilde{f}^{\mathrm{an}}_*\,(\tilde{j}_! F_{\tilde{U}})^{\mathrm{an}} \cong R(\varphi\tilde{f})^{\mathrm{an}}_*\,(\tilde{j}_! F_{\tilde{U}})^{\mathrm{an}} \cong Rf^{\mathrm{an}}_*\,(j_! F|_U)^{\mathrm{an}}
\end{aligned}$$

Here we use Thm. 9.2.2 for φ, since $R\tilde{f}(\tilde{j}_! F_{\tilde{U}})$ is constructible (cf.[LM14]).
In addition, Thm. 9.2.4 holds for $f \circ i\colon Z \to S$ by the Induction hypothesis, since $\dim(Z) < \dim(X)$. As $F|_Z$ is constant, this implies

$$(Rf_* Ri_*\, F|_Z)^{\mathrm{an}} \cong (R(fi)_* F|_Z)^{\mathrm{an}} \cong R(fi)^{\mathrm{an}}_*\,(F|_Z)^{\mathrm{an}} \cong Rf^{\mathrm{an}}_* Ri^{\mathrm{an}}_*(F|_Z)^{\mathrm{an}}$$

In particular, this means $(Rf_*(i_* F|_Z))^{\mathrm{an}} \cong Rf^{\mathrm{an}}_*(i_* F|_Z)^{\mathrm{an}}$. Finally, by the short exact sequence of Prop. 2.5.6

$$0 \to j_! F|_U \to F \to i_* F|_Z \to 0$$

the functor (-)$^{\mathrm{an}}$ yields a morphism of distinguished triangles, coinciding with the diagram above except that the symbols $\sim$ are omitted. From this diagram one concludes that

$$(Rf_*\,F)^{\mathrm{an}} \cong Rf^{\mathrm{an}}_* F^{\mathrm{an}}$$

since the vertical arrows for $Rf_*\,(j_! F|_U)$ and $Rf_*\,(i_* F|_Z)$ are isomorphisms, as outlined above.

3.2. Main Proof for $\mathrm{char}(k) = 0$

In view of the Reductions 1.,2. and 3.1 one may restrict to the case where $f\colon X \to S$ is an dense open immersion, S a smooth scheme and $F \in \mathrm{Shv}(X; \mathrm{Ab})$ constant, without affecting the generality of the statement.

Let $l(X,S)$ be the length of a maximal sequence of strict inclusions of open subschemes

$$X = X_0 \subsetneq X_1 \subsetneq X_2 \subsetneq \dots \subsetneq X_{l(X,S)} = S$$

such that each $X_{i+1} \setminus X_i$ is the dense open subscheme containing all regular points of $(S \setminus X_i)_{\mathrm{red}}$. The case $l(X,S) = 0$ is trivial. Also, if $l(X,S) = 1$, Cor. 9.2.3 provides the desired statement. This allows to proceed with an induction on $l(X,S) \geq 2$.
Let X_1 be the first open subscheme in a chain defining $l(X,S)$ and let $h\colon X_1 \to S$ be the associated open immersion. Set $Z := X_1 \setminus X$ and denote the inclusion by $i\colon Z \hookrightarrow X_1$. Then, the distinguished triangle $\triangle$ of Rmk. 9.2.5 with $G = j_*F$ reads as

$$i_* R\mathcal{H}_Z(j_*F) \longrightarrow j_*F[0] \longrightarrow Rj_*F \longrightarrow i_*R\mathcal{H}_Z(F)[1]$$

since $j^*j_*F \cong F$. Applying Rh_* and (-)$^{\mathrm{an}}$ to this triangle produces the following morphism of distinguished triangles

$$\begin{tikzcd}
(Rh_*i_*R\mathcal{H}_Z(j_*F))^{\mathrm{an}} \arrow[r] \arrow[d] & (Rh_*j_*F)^{\mathrm{an}} \arrow[r] \arrow[d] & (Rf_*F)^{\mathrm{an}} \arrow[r] \arrow[d] & (Rh_*i_*R\mathcal{H}_Z(j_*F))^{\mathrm{an}}[1] \arrow[d] \\
Rh^{\mathrm{an}}_* i^{\mathrm{an}}_* R\mathcal{H}^{\mathrm{an}}_Z(j^{\mathrm{an}}_*F) \arrow[r] & Rh^{\mathrm{an}}_* j^{\mathrm{an}}_* F^{\mathrm{an}} \arrow[r] & Rf^{\mathrm{an}}_* F^{\mathrm{an}} \arrow[r] & Rf^{\mathrm{an}}_* i^{\mathrm{an}}_* R\mathcal{H}^{\mathrm{an}}_Z(j^{\mathrm{an}}_*F)[1]
\end{tikzcd}$$

By assumption F and therefore F^{an} are constant. Thus, j_*F and $j^{\mathrm{an}}_*F^{\mathrm{an}}$ are simply the constant sheaves on X_1 with the same value as F. Thus, $(j_*F)^{\mathrm{an}} \cong j^{\mathrm{an}}_*F^{\mathrm{an}}$. As $l(X_1,S) = l(X,S) - 1$, the induction hypothesis applies to h, yielding

$$Rh_*(j_*F))^{\mathrm{an}} \cong Rh^{\mathrm{an}}_* j^{\mathrm{an}}_* F^{\mathrm{an}}.$$

By construction, the morphism $h \circ i\colon Z \to S$ factors as

$$\begin{tikzcd}
Z \arrow[r, "\eta"] \arrow[d, "i"'] & (S \setminus X)_{\mathrm{red}} \arrow[d, "\iota"] \\
X_1 \arrow[r, "h"] & S
\end{tikzcd}$$

where η exhibits Z as the dense open subscheme of regular points in $(S \setminus X)_{\mathrm{red}}$. By reduction 3.1 we may assume, that Thm. 9.2.4 applies to η. Moreover, $\dim(S \setminus X)_{\mathrm{red}}) < \dim S = \dim X$, since $X \overset{\circ}{\longrightarrow} S$ is dense. Thus, the induction hypothesis applies to $\iota\colon (X \setminus S)_{\mathrm{red}} \overset{+}{\longrightarrow} S$. Combining both of these facts yields the isomorphism

$$(Rh_*i_*\mathcal{H}_Z(j_*F))^{\mathrm{an}} \cong (R\eta_*R\iota_*\mathcal{H}_Z(j_*F))^{\mathrm{an}} \cong R\eta^{\mathrm{an}}_*R\iota^{\mathrm{an}}_*(\mathcal{H}_Z(j_*F))^{\mathrm{an}} \cong Rh^{\mathrm{an}}i^{\mathrm{an}}_*(\mathcal{H}_Z(j_*F))^{\mathrm{an}}$$

Finally, from the morphism of distinguished triangles above, it follows that
$(Rf_*F)^{\mathrm{an}} \to Rf^{\mathrm{an}}_*F^{\mathrm{an}}$ is an isomorphism, for instance by applying the five lemma to the long exact sequence associated to the triangles.

4. Case: $\mathrm{char}(k) \neq 0$
To prove the induction step in this case, we still assume $\dim(X) = d$ for some $d \geq 1$.

4.1 Reduce to X being a smooth scheme
Suppose, that Thm. 9.2.4 holds for any morphism $Y \to T$ in $\mathrm{Sch}^{\mathrm{lft}}_{/k}$ where Y is smooth.

Let $j\colon U \mathrel{-\!\!\circ\!\!\!\rightarrow} X$ be the dense open subscheme of non-singular points in X and let $Z := (X \setminus U)_{\mathrm{red}} \overset{i}{\hookrightarrow} X$ be the complement. Recall that $(\text{-})^{\mathrm{an}}$ is exact. Thus, applying Rf_* to the distinguished triangle $\triangle$ of Rmk. 9.2.5(with $G = F$), we get the following morphism of distinguished triangles

$$\begin{tikzcd}
(Rf_* i_* R\mathcal{H}_Z(F))^{\mathrm{an}} \arrow[r] \arrow[d] & (Rf_* F)^{\mathrm{an}} \arrow[r] \arrow[d] & (Rf_* Rj_* j^* F)^{\mathrm{an}} \arrow[r] \arrow[d] & (Rf_* i_* R\mathcal{H}_Z(F))^{\mathrm{an}}[1] \arrow[d] \\
Rf_*^{\mathrm{an}} i_*^{\mathrm{an}} R\mathcal{H}_Z^{\mathrm{an}}(F) \arrow[r] & Rf_*^{\mathrm{an}} F^{\mathrm{an}} \arrow[r] & Rf_*^{\mathrm{an}} Rj_*^{\mathrm{an}} j^{\mathrm{an}*} F^{\mathrm{an}} \arrow[r] & Rf_*^{\mathrm{an}} \mathcal{H}_Z^{\mathrm{an}}(F)[1]
\end{tikzcd}$$

By assumption, Thm. 9.2.4 holds for $f \circ j\colon U \to S$, since U is smooth. Thus, the middle vertical arrow is an equivalence. Now, $\operatorname{codim}(X, Z) \geq 1$, since U is dense in X. This implies $\dim Z < \dim X$. By the induction hypothesis we may apply Thm. 9.2.4 in case of $f \circ i\colon Z \to S$ and $R\mathcal{H}_Z(F)$ yielding

$$(Rf_* i_* R\mathcal{H}_Z(F))^{\mathrm{an}} \simeq (R(f \circ i)_* R\mathcal{H}_Z(F))^{\mathrm{an}} \simeq R(f \circ i)_*^{\mathrm{an}} R\mathcal{H}_Z^{\mathrm{an}}(F^{\mathrm{an}}) \simeq Rf^{\mathrm{an}} i_*^{\mathrm{an}} R\mathcal{H}_Z^{\mathrm{an}}(F).$$

In conclusion, the left vertical arrow in the diagram above is an equivalence in $D^+(S^{\mathrm{an}})$, which implies $(Rf_*F)^{\mathrm{an}} \simeq Rf_*^{\mathrm{an}} F^{\mathrm{an}}$, by the five lemma and the associated long exact sequence.

4.2 Reduce to S proper k-scheme.

As the question is Zariski-local on S, we may assume $S = \operatorname{Spec}(A)$ for a finite type k-algebra A. Let $\operatorname{Spec}(A) \mathrel{-\!\!\!+\!\!\!\rightarrow} \mathbb{A}_k^m$ be the canonical closed immersion and $\operatorname{Spec}(A) \to \mathbb{P}_k^m$ the associated immersion, induced by a choice of open embedding $\mathbb{A}_k^m \mathrel{-\!\!\circ\!\!\!\rightarrow} \mathbb{P}_k^m$. Let $\overline{S} \mathrel{-\!\!\!+\!\!\!\rightarrow} \mathbb{P}_k^m$ be the closure of the image of S in $\mathbb{P}_k^m$. The associated morphism $S \to \overline{S}$ is a dense open immersion. Suppose Thm. 9.2.4 holds for the dense open immersion $X \overset{f}{\mathrel{-\!\!\circ\!\!\!\rightarrow}} S \mathrel{-\!\!\circ\!\!\!\rightarrow} \overline{S}$. Again, using the distinguished triangle $\triangle$ of Rmk. 9.2.5 and running the exact same argument as in step **4.1**, this implies $(Rf_*F)^{\mathrm{an}} \simeq Rf_*^{\mathrm{an}} F^{\mathrm{an}}$. Thus, we may replace S by $\overline{S}$, which per Definition is a projective k-scheme. Next, we prove the following

4.3. Claim *There is an open subset $V \subset S$, such that the induction hypothesis applies to $f_{X \cap V}\colon X \cap V \to V$ and $S \setminus V$ is a finite k-scheme.*

Again, we may take $S = \operatorname{Spec}(A)$ for a finite type k-algebra A. The canonical closed immersion $S \mathrel{-\!\!\!+\!\!\!\rightarrow} \mathbb{A}_k^n$ gives rise to projections $\pi_i\colon S \to \mathbb{A}_k^1$ for $i = 1, .., m$. Let η be the generic point of $\mathbb{A}_k^1$ and let $Z \subset S_\eta$ be an irreducible, closed subset of the generic fiber. The induced map $Z \to \mathbb{A}_k^1$ is a finite type, dominant morphism of integral schemes. Therefore, $\dim(Z)$ is given by the transcendence degree $\operatorname{trdeg}_{K(T)} K(Z)$ of the extension of function fields $K(Z)/K(T)$, where $K(T) = K(\mathbb{A}_k^1)$. Since $\dim S \geq \operatorname{trdeg}_k K(Z)$, this implies $\dim Z = \operatorname{trdeg}_k K(Z) - 1 \leq \dim S - 1$. In conclusion, $\dim S_\eta = \dim S - 1$, which implies $\dim X_\eta = \dim X - 1$, since $X_\eta \mathrel{-\!\!\circ\!\!\!\rightarrow} S_\eta$ is dense. By the upper semi-continuity of fiber dimension [FOAG, Theorem 12.4.3], there exists an open neighbourhood U_i of η, such that $\dim(X \cap \pi_i^{-1} U_i) = \dim X - 1$. Thus, the induction hypothesis applies to $f|_{X \cap \pi_i^{-1} U_i}\colon X \cap \pi_i^{-1} U \mathrel{-\!\!\circ\!\!\!\rightarrow} S$ and in particular to $f|_{X \cap V}\colon X \cap V \mathrel{-\!\!\circ\!\!\!\rightarrow} V$ for $V := \cup_{i=1}^m \pi_i^{-1} U_i$.

4.4 Main Proof for $\operatorname{char}(k) \neq 0$

Utilizing the previous steps, there is a commutative diagram

$$\begin{tikzcd}
X \cap V \arrow[r, "f_{|X \cap V}"] & V & \\
\cap & \cap & \\
X \arrow[r, "f"] \arrow[dr, "g"'] & S \arrow[d, "s"] & Z := S \setminus V \arrow[l, "i"] \arrow[dl, "t"] \\
& k &
\end{tikzcd}$$

such that g is smooth, s is proper, t is finite and the induction hypothesis applies to $f|_{X \cap V}$. For any sheaf $G \in \mathrm{Shv}_{\acute{e}t}(S, \mathrm{Ab})$ consider the short exact sequence of Prop. 2.5.6

$$0 \longrightarrow j_! G|_V \longrightarrow G \longrightarrow i_* G|_Z \longrightarrow 0.$$

Recall from Prop. 2.5.6 and Rmk. 9.2.5 that the functors $j_!$ and i_* are exact. Thus, setting $G = R^q f_* F$, the sequence above gives rise to a short exact sequence of cochain complexes

$$0 \longrightarrow j_! j^* Rf_* F \longrightarrow Rf_* F \longrightarrow i_* i^* Rf_* F \longrightarrow 0. \qquad (\boxed{\varsigma})$$

Applying (-)$^{\mathrm{an}}$, which is exact, to this sequence, yields a morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & (j_! j^* Rf_* F)^{\mathrm{an}} \arrow[r] \arrow[d] & (Rf_* F)^{\mathrm{an}} \arrow[r] \arrow[d] & (i_* i^* Rf_* F)^{\mathrm{an}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & j_!^{\mathrm{an}} (j^{\mathrm{an}})^* Rf_*^{\mathrm{an}} F^{\mathrm{an}} \arrow[r] & Rf_*^{\mathrm{an}} F^{\mathrm{an}} \arrow[r] & i_*^{\mathrm{an}} (i^{\mathrm{an}})^* Rf_*^{\mathrm{an}} F^{\mathrm{an}} \arrow[r] & 0
\end{tikzcd} \qquad (\boxed{\mu})$$

Since S is proper, the open immersion f is its own compactification i.e. $Rf_! \cong Rf_*$ in the notation of Def. 8.1.5. Since f^{an} and j^{an} are open immersions of locally Hausdorff spaces, they are restricted and Hausdorff, in particular. Applying Smooth Base Change [Sta25, Tag 0EYU], Restricted Base Change 8.1.10 and the induction hypothesis (I.H.), one concludes

$$(j_! j^* Rf_* F)^{\mathrm{an}} \overset{\text{[Sta25, Tag 0EYU]}}{\simeq} (j_! Rf_* j^* F)^{\mathrm{an}} \overset{I.H.}{\simeq} j_!^{\mathrm{an}} Rf_*^{\mathrm{an}} (j^{\mathrm{an}})^* F^{\mathrm{an}} \overset{8.1.10}{\simeq} j_!^{\mathrm{an}} (j^{\mathrm{an}})^* Rf_*^{\mathrm{an}} F^{\mathrm{an}}.$$

To verify that the third vertical arrow is an isomorphism as well, one applies Rs_* to the short exact sequence $\boxed{\varsigma}$, thereby obtaining a distinguished triangle. Applying (-)$^{\mathrm{an}}$ to this triangle again, yields the following morphism of distinguished triangles.

$$\begin{tikzcd}
0 \arrow[r] & (Rs_* j_! j^* Rf_* F)^{\mathrm{an}} \arrow[r] \arrow[d] & (Rg_* F)^{\mathrm{an}} \arrow[r] \arrow[d] & (Rt_* i^* Rf_* F)^{\mathrm{an}} \arrow[r] \arrow[d] & (Rs_* j_! j^* Rf_* F)^{\mathrm{an}}[1] \arrow[d] \\
0 \arrow[r] & Rs_*^{\mathrm{an}} j_!^{\mathrm{an}} (j^{\mathrm{an}})^* Rf_*^{\mathrm{an}} F^{\mathrm{an}} \arrow[r] & Rg_*^{\mathrm{an}} F^{\mathrm{an}} \arrow[r] & Rt_*^{\mathrm{an}} (i^{\mathrm{an}})^* Rf_*^{\mathrm{an}} F^{\mathrm{an}} \arrow[r] & Rs_*^{\mathrm{an}} j_!^{\mathrm{an}} (j^{\mathrm{an}})^* Rf_*^{\mathrm{an}} F^{\mathrm{an}}[1]
\end{tikzcd}$$

Now, $(Rg_* F)^{\mathrm{an}} \simeq Rg_*^{\mathrm{an}} F^{\mathrm{an}}$ by Cor. 9.2.3, since g is smooth. Thus, the five lemma implies

$$(Rt_* i^* Rf_* F)^{\mathrm{an}} \simeq Rt_*^{\mathrm{an}} (i^{\mathrm{an}})^* Rf_*^{\mathrm{an}} F^{\mathrm{an}}.$$

Let $K := \widehat{k}^a$ be an algebraic closure. Let $\bar{x} \colon \mathrm{Spec}(K) \to \mathrm{Spec}(k)$ and $\bar{x}^{\mathrm{an}}$ denote the corresponding geometric points. Since $t \colon Z \to \mathrm{Spec}(k)$ is finite, t^{an} is finite as well (cf. 6.3.5). The

Finite Vanishing Theorems 8.1.8 and [Sta25, Tag 03QP] imply

$$\begin{aligned} (Rt_*i^*Rf_*F)_{\bar{x}} &\simeq (t_*i^*Rf_*F)_{\bar{x}}[0] &&\simeq \bigoplus_{\bar{z}\colon \operatorname{Spec}(K)\to Z \text{ s.t. } t\circ\bar{z}=\bar{x}} (i^*Rf_*F)_{\bar{z}}[0] \\ (Rt_*^{\mathrm{an}}(i^*)^{\mathrm{an}}Rf_*^{\mathrm{an}})_{\bar{x}^{\mathrm{an}}} &\simeq (t_*^{\mathrm{an}}(i^*)^{\mathrm{an}}Rf_*^{\mathrm{an}}F^{\mathrm{an}})_{\bar{x}^{\mathrm{an}}}[0] &&\simeq \bigoplus_{\bar{\zeta}\colon \mathcal{M}(K)\to Z^{\mathrm{an}} \text{ s.t. } t^{\mathrm{an}}\circ\zeta=\bar{x}^{\mathrm{an}}} ((i^*)^{\mathrm{an}}Rf_*^{\mathrm{an}}F^{\mathrm{an}})_{\bar{\zeta}}[0]. \end{aligned}$$

Every geometric point of $\mathcal{M}(k)$ or Z^{an} is the analytification of a geometric point of $\operatorname{Spec}(k)$ or Z (cf. 7.3.38). Therefore, the three equivalences combine to

$$\bigoplus_{\bar{\zeta}\colon \mathcal{M}(K)\to Z^{\mathrm{an}} \text{ s.t. } t\circ\bar{\zeta}=\bar{x}^{\mathrm{an}}} (i^*Rf_*F)_{\bar{\zeta}}^{\mathrm{an}} \simeq (Rt_*i^*Rf_*F)_{\bar{x}^{\mathrm{an}}}^{\mathrm{an}} \simeq (Rt_*^{\mathrm{an}}(i^{\mathrm{an}})^*Rf_*^{\mathrm{an}}F^{\mathrm{an}})_{\bar{x}^{\mathrm{an}}} \simeq \bigoplus_{\bar{\zeta}\colon \mathcal{M}(K)\to Z^{\mathrm{an}} \text{ s.t. } t^{\mathrm{an}}\circ\zeta=\bar{x}^{\mathrm{an}}} ((i^*)^{\mathrm{an}}Rf_*^{\mathrm{an}}F^{\mathrm{an}})_{\bar{\zeta}}.$$

for every geometric point $\bar{x}$. In particular, we have $(i^*Rf_*F)_{\bar{\zeta}}^{\mathrm{an}} \simeq ((i^*)^{\mathrm{an}}Rf_*^{\mathrm{an}}F^{\mathrm{an}})_{\bar{\zeta}}$ for every geometric point $\bar{\zeta}$ of Z^{an}. In conclusion, the quasi-isomorphism involving Rt_* becomes

$$(i^*Rf_*F)^{\mathrm{an}} \simeq (i^{\mathrm{an}})^*Rf_*^{\mathrm{an}}F^{\mathrm{an}}.$$

In particular, this implies

$$(i_*i^*Rf_*F)^{\mathrm{an}} \simeq i_*^{\mathrm{an}}(i^{\mathrm{an}})^*Rf_*^{\mathrm{an}}F^{\mathrm{an}}$$

which is the third vertical arrow in the morphism $\boxed{\mu}$. A final application of the five lemma to the long exact sequence associated to $\boxed{\mu}$ implies

$$(Rf_*F)^{\mathrm{an}} \cong Rf_*^{\mathrm{an}}F^{\mathrm{an}}. \qquad \square$$

9.3 A General Artin Comparison Theorem

As outlined in the previous sections, the analytification $(\text{-})^{\mathrm{an}}\colon \mathrm{Sch}_{k}^{\mathrm{lft}} \to \mathrm{Ber}_k$ induces a bijection of connected components (9.1.14), an equivalence on étale principal G-bundles (9.1.14) and an isomorphism on étale, abelian sheaf cohomology (9.2.4) for all truncated coherent sheaves. Thus, the conditions of Thm. 4.6.2 are satisfied and as a consequence we get the following result.

Notation 9.3.1. Let "anét" denote open set topology in context of complex analytic spaces and étale topology otherwise.

Theorem 9.3.2. *Let k be either $\mathbb{C}$ or an non-archimedean Banach field.*
*For any scheme X locally of finite type over k and any **truncated coherent** sheaf $F \in \mathrm{Shv}_{\acute{e}t}(X)$ with stalks coprime to* $\operatorname{char}(k)$*, analytification induces an equivalence*

$$\Gamma_{\mathrm{Shv}_{\acute{e}t}(X)}(F) \simeq \Gamma_{\mathrm{Shv}_{\mathrm{an}}(X^{\mathrm{an}})}(F^{\mathrm{an}})$$

Using the generalized Proper Base Change Thm. 8.0.1, we may even generalize the étale Comparison of derived higher images (Thm. 9.2.2) from the setting of torsion abelian sheaves to truncated coherent ∞-sheaves.

Theorem 9.3.3 (Proper and Smooth Comparison)**.** *Let k be either $\mathbb{C}$ or a non-archimedean complete and set $p := \operatorname{char}(\tilde{k})$. Let $f\colon X \to Y$ be a **proper** or **smooth** morphism of qcqs schemes locally of finite type over k. Let $\mathrm{Shv}_{\acute{e}t}(X)^{p,\mathrm{tc}} \subset \mathrm{Shv}_{\acute{e}t}(X)$ be the full subcategory of truncated* *coherent objects with stalks coprime to p. Then, the following diagram commutes*

$$\begin{tikzcd}
\mathrm{Shv}_{\textit{ét}}(X)^{p,\,\mathrm{tc}} \arrow[r, "f_*"] \arrow[d, "(-)^{\mathrm{an}}"] & \mathrm{Shv}_{\textit{ét}}(Y) \arrow[d, "(-)^{\mathrm{an}}"] \\
\mathrm{Shv}_{\textit{anét}}(X^{\mathrm{an}}) \arrow[r, "f_*^{\mathrm{an}}"] & \mathrm{Shv}_{\textit{anét}}(Y^{\mathrm{an}})
\end{tikzcd}$$

Put differently, analytification induces an equivalence for all ***truncated coherent*** $F \in \mathrm{Shv}_{\textit{ét}}(X)$ *with stalks coprime to p:*

$$(f_*F)^{\mathrm{an}} \xrightarrow{\simeq} f_*^{\mathrm{an}} F^{\mathrm{an}} \tag{$\spadesuit$}$$

Remark 9.3.4. The restriction on characteristic of the sheaves only comes from Cor. 8.4.4 and might be lifted in the future.

Proof of Thm. 9.3.3. As observed in Cor. 7.2.12, the ∞-topos $\mathrm{Shv}_{\mathrm{anét}}(Y^{\mathrm{an}})$ has enough points. Thus, it suffices to check the equivalence $\spadesuit$ on each point of $\mathrm{Shv}_{\mathrm{anét}}(Y^{\mathrm{an}})$. By virtue of Thm. 7.3.36 and Cor. 7.3.37 every point of $\mathrm{Shv}_{\mathrm{anét}}(Y^{\mathrm{an}})$ is induced by a geometric point $\mathcal{M}(K) \to Y^{\mathrm{an}}$ resp. $\mathrm{Spec}(\mathbb{C}) \to Y^{\mathrm{an}}$. Let s be a geometric point of Y^{an}. Since all geometric points of Y^{an} come from geometric points of Y (see Ex. 5.0.14 and Ex. 6.3.4), we concluded in Cor. 7.3.38, that there is a unique geometric point $p\colon \mathrm{Spec}(K) \to Y$ and a commutative diagram

$$\begin{tikzcd}
\mathrm{Shv}_{\textit{ét}}(\mathrm{Spec}(K)) \arrow[r, "p_*"] \arrow[d, "(\text{-})^{\mathrm{an}}"] & \mathrm{Shv}_{\textit{ét}}(Y) \arrow[d, "(\text{-})^{\mathrm{an}}"] \\
\mathrm{Shv}_{\textit{anét}}(\mathcal{M}(K)) \arrow[r, "s_*"] & \mathrm{Shv}_{\textit{anét}}(Y^{\mathrm{an}})
\end{tikzcd}$$

i.e. $s \simeq p^{\mathrm{an}}$ in $\mathcal{RT}\mathrm{op}$.

Take pullbacks $X_K := X \times_Y \mathrm{Spec}(K)$ of f along the point p and $X_K^{\mathrm{an}} := X^{\mathrm{an}} \times_{Y^{\mathrm{an}}} \mathcal{M}(K)$ of f^{an} along p^{an}. This leaves us with the following commutative squares in $\mathcal{RT}\mathrm{op}$

$$\begin{tikzcd}
\mathrm{Shv}_{\textit{ét}}(X_K) \arrow[r, "q_*"] \arrow[d, "g_*"'] & \mathrm{Shv}_{\textit{ét}}(X) \arrow[d, "f_*"] & \mathrm{Shv}_{\textit{anét}}(X_K^{\mathrm{an}}) \arrow[r, "q_*^{\mathrm{an}}"] \arrow[d, "g_*^{\mathrm{an}}"'] & \mathrm{Shv}_{\textit{anét}}(X^{\mathrm{an}}) \arrow[d, "f^{\mathrm{an}}"] \\
\mathrm{Ani} \simeq \mathrm{Shv}_{\textit{ét}}(\mathrm{Spec}(K)) \arrow[r, "p_*"] & \mathrm{Shv}_{\textit{ét}}(Y) & \mathrm{Ani} \simeq \mathrm{Shv}_{\textit{anét}}(\mathcal{M}(K)) \arrow[r, "p_*^{\mathrm{an}}"] & \mathrm{Shv}_{\textit{anét}}(Y^{\mathrm{an}})
\end{tikzcd}$$

Since Ani is a terminal object of $\mathcal{RT}\mathrm{op}$, we may identify the left vertical arrows g_* and g_*^{an} with the global section functors $\Gamma_{\mathrm{Shv}_{\textit{ét}}(X_K)}$ and $\Gamma_{\mathrm{Shv}_{\textit{anét}}(X_K^{\mathrm{an}})}$ respectively. Finally, we conclude from the generalized Base Change Theorems 8.0.1, 8.4.6 and from Cor. 9.3.2 that $(-)^{\mathrm{an}}$ participates in the following chain of equivalences

$$\begin{aligned}
p^*f_*F &\overset{\text{Thm. 8.0.1}}{\simeq} g_*q^*F \simeq \Gamma_{\mathrm{Shv}_{\textit{ét}}(X_K)}(q^*F) \overset{\text{Cor. 9.3.2}}{\simeq} \Gamma_{\mathrm{Shv}_{\mathrm{an}}(X_K^{\mathrm{an}})}((q^*F)^{\mathrm{an}}) \\
&\simeq g_*^{\mathrm{an}}(q^*F)^{\mathrm{an}} \overset{2.4.6}{\simeq} g_*^{\mathrm{an}}(q^{\mathrm{an}})^*F^{\mathrm{an}} \overset{\text{Thm. 8.4.6}}{\simeq} (p^{\mathrm{an}})^* f_*^{\mathrm{an}} F^{\mathrm{an}}\,.
\end{aligned}$$

Here, Thm. 8.4.6 can only be applied, since F is finitely constructible because of Prop. 7.1.18 and because (-)$^{\mathrm{an}}$ preserves finite constructible objects since it is a morphism in $\mathcal{LT}\mathrm{op}$. In consequence, we have the following equivalence for any point p

$$(p^{\mathrm{an}})^*(f_*F)^{\mathrm{an}} \simeq (p^*f_*F)^{\mathrm{an}} \simeq ((p^{\mathrm{an}})^* f_*^{\mathrm{an}} F^{\mathrm{an}}$$

This implies the asserted equivalence $\spadesuit$, as $\mathrm{Shv}_{\mathrm{an}}(Y^{\mathrm{an}})$ has enough points by Cor. 7.2.12. □

Theorem 9.3.5 (Comparison)**.** *Let k be either $\mathbb{C}$ or a non-archimedean complete field. For any morphism $f\colon X \to Y$ of qcqs locally finite type k-schemes, analytification induces an equivalence for all* ***truncated coherent*** *$F \in \mathrm{Shv}_{\acute{e}t}(X)$ whose stalks are coprime to* $\mathrm{char}(\tilde{k})$

$$(f_*F)^{\mathrm{an}} \xrightarrow{\simeq} f_*^{\mathrm{an}} F^{\mathrm{an}}$$

Proof. f is locally of finite type by cancellation property of morphisms locally of finite type ([Sta25, Tag 01T8]). As the question is Zariski-local on S and X, we may assume that X is affine and f is of finite type. By the cancellation property of separated morphisms [Sta25, Tag 01KV], we may thus assume that f is separated as well. By Nagata's Theorem C.2.1, f admits a compactification i.e. a factorization

$$\begin{tikzcd}
X \arrow[rr, hook, "j"] \arrow[dr, "f"'] & & \bar{X} \arrow[dl, "\bar{f}"] \\
& Y &
\end{tikzcd}$$

where j is an open immersion and $\bar{f}$ is proper. Applying Thm. 9.3.3 twice yields

$$(f_*F)^{\mathrm{an}} \simeq (\bar{f}_* j_* F)^{\mathrm{an}} \overset{9.3.3}{\simeq} \bar{f}_*^{\mathrm{an}} j_*^{\mathrm{an}} F^{\mathrm{an}} \simeq f_*^{\mathrm{an}} F^{\mathrm{an}}.$$ □

Corollary 9.3.6. *Due to the results in § 2.11, the statements of Thm. 9.3.2 and Thm. 9.3.5 hold for all sheaves with values in any presentable category.*

9.4 A Comparison of étale Homotopy Types

Since the analytification of schemes locally of finite type induces sufficient comparisons of connected components, principal bundles and abelian cohomology of constant sheaves, Thm. 4.6.4 provides the following special case of Cor. 9.3.2

Theorem 9.4.1. *Let k be a non-archimedean Banach field and set $p :=$ $\mathrm{char}(k)$. Then,* $(\text{-})^{\mathrm{an}}$ *induces the following equivalences of (p)-profinite shape functors.*

$$\widehat{\Pi}_{\infty}^{\mathrm{Shv}_{\mathrm{open}}(\mathrm{AnSp}_{\mathbb{C}})} \circ (\text{-})^{\mathrm{an}} \overset{\simeq}{\Longrightarrow} \widehat{\Pi}_{\infty}^{\mathrm{Shv}_{\acute{e}t}(\mathrm{Sch}^{\mathrm{lft}}_{/\mathbb{C}})} \qquad {}^{p}\widehat{\Pi}_{\infty}^{\mathrm{Shv}_{an\acute{e}t}(\mathrm{Ber}_k)} \circ (\text{-})^{\mathrm{an}} \overset{\simeq}{\Longrightarrow} {}^{p}\widehat{\Pi}_{\infty}^{\mathrm{Shv}_{\acute{e}t}(\mathrm{Sch}^{\mathrm{lft}}_{/k})}$$

The first equivalence turns out to be particularly interesting, as every analytic space locally admits a triangulation, thus is a locally contractible space (cf.[Loj64] or [Gie64]). Thus, by Lem. 3.2.16, $\mathrm{Shv}_{open}(\mathrm{AnSp}_{\mathbb{C}})^{\mathrm{hyp}}$ is locally ∞-connected, since $\mathrm{Shv}_{\mathrm{open}}(V)^{\mathrm{hyp}}$ is locally ∞-connected for every $V \in \mathrm{AnSp}_{\mathbb{C}}$ by Cor. 3.2.17. Therefore, Lem. 3.2.8 induces the following specialization.

Corollary 9.4.2 ([Car16, 4.13])**.** *The shape of the ∞-topos of étale sheaves over $\mathrm{Sch}^{\mathrm{lft}}_{/\mathbb{C}}$ is constant and given by*

$$\widehat{(-)} \circ \Pi_{\mathrm{const}}^{\mathrm{Shv}_{\mathrm{open}}(\mathrm{AnSp}^{\mathrm{hyp}})} \circ \mathfrak{h}\mathrm{yp} \circ (\text{-})^{\mathrm{an}} \xRightarrow{\simeq} \widehat{\Pi}_{\infty}^{\mathrm{Shv}_{\acute{e}t}(\mathrm{Sch}^{\mathrm{lft}}_{\mathbb{C}})}.$$

Equivalently, following diagram commutes

$$\begin{tikzcd}
\mathrm{Shv}_{\acute{e}t}(\mathrm{Sch}^{\mathrm{lft}}_{\mathbb{C}}) \arrow[rr, "\widehat{\Pi}_{\infty}^{\mathrm{Shv}_{\acute{e}t}(\mathrm{Sch}^{\mathrm{lft}}_{\mathbb{C}})}"] \arrow[d, "(\text{-})^{\mathrm{an}}"] & & \mathrm{Pro}(\mathrm{Ani}^{\pi}) \\
\mathrm{Shv}_{\mathrm{open}}(\mathrm{AnSp}_{\mathbb{C}}) \arrow[r, "\mathfrak{h}\mathrm{yp}"] & \mathrm{Shv}_{\mathrm{open}}(\mathrm{AnSp}_{\mathbb{C}})^{\mathrm{hyp}} \arrow[r, "\Pi_{\mathrm{const}}^{\mathrm{Shv}_{\mathrm{open}}(\mathrm{AnSp}_{\mathbb{C}})^{\mathrm{hyp}}}"] & \mathrm{Ani} \arrow[u, "\widehat{(-)} = i_{\pi}^{*} \circ j"']
\end{tikzcd}$$

In particular, the profinite étale shape of a scheme $X \in \mathrm{Sch}^{\mathrm{lft}}_{/\mathbb{C}}$ is given by the profinite space

$$\Pi_{\infty} \mathrm{Shv}_{\acute{e}t}(X) \simeq \widehat{\mathrm{Sing}(X^{\mathrm{an}})}.$$

This way various classical results are recovered

Example 9.4.3 ([Car16, 4.15])**.** Let $\mathcal{M}_{g,n}$ denote the moduli stack proper smooth curves of genus g with n marked points. Classically, $\mathcal{M}_{g,n}$ is defined as the 1-category, whose objects are triples (S, p, Σ), where S is a scheme, $p\colon C \to S$ is a proper smooth morphism, whose geometric fibres are connected algebraic curves of genus g and $\Sigma = (\sigma_1, ..., \sigma_n)$ is a n-tuple of sections $\sigma_i\colon S \to C$ of p, such that $\sigma_i(s) \neq \sigma_j(s)$ for all $s \in S$ whenever $i \neq j$. A morphism $(S, p, \Sigma) \to (S', p', \Sigma')$ is a pullback square

$$\begin{tikzcd}
C \arrow[r, "\alpha"] \arrow[d, "p"] & C' \arrow[d, "p'"] \\
S \arrow[r, "\beta"] & S'
\end{tikzcd}$$

which is compatible with the sections. The forgetful functor

$$f\colon \mathcal{M}_{g,n} \to \mathrm{Sch}, \qquad (S, p, \Sigma) \mapsto S$$

is cartesian a fibration in the sense of [Lan21, §3] (in classical terms, $\mathcal{M}_{g,n}$ is a category fibred in groupoids). By definition, the fiber F_S is the 1-groupoid of smooth proper curves of genus g over S with n fixed points. Under the straightening equivalence [Lan21, 3.3.10], F corresponds to the presheaf

$$F := \mathrm{Str}(f)\colon \mathrm{Sch}^{\mathrm{op}} \to \tau_{\leq 1}\mathrm{Ani}.$$

The fact that $\mathcal{M}_{g,n}$ forms an étale stack (cf.[DM69]) implies that F i.e. $F \in \mathrm{Shv}_{\acute{e}t}$ is an étale sheaf with values in groupoids.

Consider the restriction of this sheaf $F_{/\mathbb{C}}\colon (\mathrm{Sch}^{\mathrm{lft}}_{/k})^{\mathrm{op}} \to \tau_{\leq 1}\mathrm{Ani}$.
Oda [Oda97, Theorem 4] showed, that the analytification $F^{\mathrm{an}}_{/\mathbb{C}}$ admits an effective epimorphism from the Teichmüller space $y(T_{g,n}) \to F^{\mathrm{an}}_{\mathbb{C}}$. Since the homotopy type of this space is equivalent to the classifying space $\mathbf{B}\Gamma_{g,n}$ of the mapping class group $\Gamma_{g,n}$ of a surface of genus g with n

marked points, one concludes

$$\widehat{\Pi}_\infty^{\mathrm{Shv}_{\acute{e}t}(\mathrm{Sch}^{\mathrm{lft}}_{/\mathbb{C}})} \mathcal{M}_{g,n} \simeq \widehat{\mathbf{B}\Gamma_{g,n}}.$$

Example 9.4.4 ([Car16, 4.16])**.** Let $\mathcal{M}_{\mathrm{ell}}$ denote the moduli stack of elliptic curves. Frediani and Neumann [FN16] showed, that for any embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, the homotopy type of the analytification is $(\mathcal{M}_{\mathrm{ell}} \otimes \overline{\mathbb{Q}})^{\mathrm{an}} \simeq \mathbf{B}\,\mathrm{SL}(2,\mathbb{Z})$. In consequence,

$$\widehat{\Pi}_\infty \,\mathrm{Shv}_{\mathrm{\acute{e}t}}(\mathcal{M}_{\mathrm{ell}} \otimes \overline{\mathbb{Q}}) \simeq \mathbf{B}\,\widehat{\mathrm{SL}(2,\mathbb{Z}})).$$

10 A Comparison of Shapes over Non-Archimedean Fields

One of the most significant properties of Berkovich geometry is that many of its are locally contractible, providing both a bug and a feature to the theory. In contrast to classical rigid geometry, the underlying topological space of the affine line $|\mathbb{A}n^1_k|$ is contractible, exhibiting Berkovich analytification as a natural tool for studies in $\mathbb{A}^1$-homotopical context (cf.[Vez19; MV24]). However, for some applications Berkovich spaces can be even too contractible to gather relevant information (cf.[Duc11]). However, as we have seen in the previous section, locally contractible spaces give rise to locally ∞-connected topoi and thus make the (profinite) shape of an ∞-topos computable, given an suitable comparison of classical (non)-abelian cohomology groups. After recalling the classical results giving rise to the contractiblity of many Berkovich spaces, we will see that the cdh-topology compares especially well to Berkovich geometry, examinig various comparisons along the way.

Convention 10.0.1. For this section, let k be a Non-Archimedean valued complete field.

10.1 Contractibility of Analytic Spaces

In his famous papers [Ber99a] and [Ber04], Berkovich established that smooth k-analytic spaces are locally contractible using the theory of formal models. Advancing this framework further Hrushovski and Loeser [HL16] show that for every quasi-projective algebraic k-variety retracts on a finite simplicial complex and thus, in particular, is locally contractible. Another statement comes from the relation of k-analytic spaces to formal schemes. Set $k^\circ := \{x \in k \mid |x| \leq 1\}$. As outlined in [Tem11, §5.2], one may functorially associate to every formal scheme $\mathfrak{X}$ locally of finite type over k° a Hausdorff strictly k-analytic space $\mathfrak{X}_\eta$, called the generic fibre of $\mathfrak{X}$. As commonly known, the resulting functor coincides with $(\text{-})^{\mathrm{an}}$ on proper schemes if $k = k^\circ$.

Proposition 10.1.1 ([Tem11, 5.2.3.1])**.** *Let k be equipped with the trivial valuation, the functors*

$$(\text{-})^{\beth}\colon \mathrm{Sch}^{\mathrm{lft}}_{/k} \xrightarrow{\widehat{(\text{-})}} \mathrm{FormSch}^{\mathrm{lft}}_{/k} \xrightarrow{(\text{-})_\eta} \textit{and} \qquad (\text{-})^{\mathrm{an}}\colon \mathrm{Sch}^{\mathrm{lft}}_{/k} \to \mathrm{Ber}_k$$

are equivalent when restricted to the category of proper locally finite type k-schemes.

Molokov and Vologodsky [MV24] propose that Thuillier's work on formal schemes and toroidal embeddings implies the following result.

Proposition 10.1.2 ([Thu07, 3.26], [MV24, 3.20])**.** *Let k carry the trivial valuation. If X is an open subscheme of a proper smooth locally finite type k-scheme, then $X^{\beth} \cong X^{\mathrm{an}}$ is contractible.*

Corollary 10.1.3. *Suppose k allows (strong) resolutions of singularities (C.1.5). For any smooth separated scheme X locally of finite type over k, the k-analytic space X^{an} is contractible, when k is equipped with the trivial valuation.*

Proof. Since k admits resolutions of singularities, there exists a smooth compactification Y of X, that is a proper smooth locally finite type k-scheme Y and an open immersion $X \hookrightarrow Y$ (cf. C.2.3). Since X^{an} is an open subspace of the proper smooth k-analytic space Y^{an}, it is contractible. ☐

The goal of this section is to obtain a comparison of shapes similar to Thm. 9.4.1 for the case of cdh and Berkovich analytic topology. Due to the contractiblity results above, the shape of many k-analytic spaces is constant giving rise to similar results as Cor. 9.4.2.

Notation 10.1.4. $\mathrm{Sch}_{/k}$ denotes the category of schemes over k and $\mathrm{Sm}_{/k} \subset \mathrm{Sch}_{/k}$ the full subcategory of smooth schemes of finite type over k.

10.2 Comparison of Zariski and Nisnevich topology

Let us set the stage for the first comparison.

Definition 10.2.1. Let $S_{\mathbb{A}^1}$ be the class of morphisms $y(X \times_k \mathbb{A}^1_k) \to y(X)$ in $\mathcal{P}(\mathrm{Sm}_{/k})$ induced by the projection $X \times_k \mathbb{A}^1_k \to X$ for some $X \in \mathrm{Sm}_{/k}$. Let

$$L_{\mathbb{A}^1} \colon \mathcal{P}(\mathrm{Sm}_{/k}) \to \mathcal{P}(\mathrm{Sm}_{/k})[S^{-1}_{\mathbb{A}^1_k}]$$

be the localization[76] along $S_{\mathbb{A}^1}$. A presheaf $F \in \mathcal{P}(\mathrm{Sm}_{/k})$ is said to be **homotopy invariant**, if it lies in the image of the inclusion $\mathcal{P}(\mathrm{Sm}_{/k})[S^{-1}_{\mathbb{A}^1}] \subset \mathcal{P}(\mathrm{Sm}_{/k})$.

Remark 10.2.2. Let $\mathcal{D}$ be a presentable category. In view of § 2.11 this terminology is extended to presheaves with values in $\mathcal{D}$, since tensoring with $\mathcal{D}$ preserves morphisms in Pr^L, thus yields the localization

$$L_{\mathbb{A}^1} \colon \mathcal{P}(\mathrm{Sm}_{/k}; \mathcal{D}) \to \mathcal{P}(\mathrm{Sm}_{/k}; \mathcal{D})[S^{-1}_{\mathbb{A}^1_k}]$$

Therefore, a presheaf $F \in \mathcal{P}(\mathrm{Sm}_{/k}; \mathcal{D})$ admits transfers, if it lies in $\mathcal{P}(\mathrm{Sm}_{/k}; \mathcal{D})[S^{-1}_{\mathbb{A}^1}]$.

Convention 10.2.3. For the remainder of this section, fix an arbitrary, ∞-category $\mathcal{C}$ with an initial object $\emptyset \in \mathcal{C}$.

Definition 10.2.4. A **cd-structure** on $\mathcal{C}$ is a (possibly empty) class $\mathcal{Q}$ of commutative squares in $\mathcal{C}$, which is closed under equivalences. The elements of $\mathcal{Q}$ are called $\mathcal{Q}$-distinguished squares.

A **cd-structure** on a 1-category C with initial object $\emptyset_C$ is a class Q of commutative diagrams in C, that is closed under isomorphisms. Thus, Q is a cd-structure if and only if it specifies a cd-structure in $\mathrm{N}(C)$.

Observation 10.2.5. Unions and intersections of cd-structures are again cd-structures.

Definition 10.2.6. For a cd-structure $\mathcal{Q}$ on $\mathcal{C}$, one associates a Grothendieck topology $\tau_{\mathcal{Q}}$ to $\mathcal{Q}$, by defining $\tau_{\mathcal{Q}}$ to be the coarsest Grothendieck topology, such that

1) the empty sieve covers the initial object $\emptyset_{\mathcal{C}}$

2) $\{v \to x, w \to x\}$ generates a covering sieve for every $\mathcal{Q}$-distinguished square

$$\begin{tikzcd}
u \arrow[r] \arrow[d] & v \arrow[d] \\
w \arrow[r] & x
\end{tikzcd}$$

---

[76] i.e. $\mathcal{P}(\mathrm{Sm}_{/k}[S^{-1}_{\mathbb{A}^1_k}] \subset \mathcal{P}(\mathrm{Sm}_{/k})$ is the full subcategory of $S_{\mathbb{A}^1}$-local objects and $L_{\mathbb{A}^1}$ is the reflection of this subcategory, obtained by localizing along the strongly saturated closure of $S_{\mathbb{A}^1}$ (cf. [HTT, 5.5.4.15].

Example 10.2.7. The Zariski topology admits a description as cd-structure: Let Q_{Zar} be the collection of squares

$$\begin{tikzcd}
W \arrow[r] \arrow[d] & V \arrow[d] \\
U \arrow[r] & X
\end{tikzcd}$$

where $U \to X$ and $V \to X$ are open immersions, such that $U \coprod V \to X$ is surjective and $W \cong U \times_X V$. By Definition, the covering sieve generated by $\{U \to X, V \to X\}$ is a Zariski covering sieve and likewise every Zariski covering generated by $\{U_i \to X\}_{i \in I}$ for $|I| > 1$ is contained in the Q_{Zar}-sieve $\{U_j \to X, \cup_{i \neq j} U_i \to X\}$ for some $j \in I$. In case $|I| = 1$, the sieve in question is the trivial sieve.

Definition 10.2.8. A **elementary Nisnevich square** is a pullback square in Sch_k

$$\begin{tikzcd}
W \arrow[r] \arrow[d] & V \arrow[d] \\
U \arrow[r] & X
\end{tikzcd}$$

such that $U \to X$ is an open immersion, $V \to X$ is étale and $(V - W) \to (X - U)$ is an isomorphism. Define the **Nisnevich cd-structure** to be the class of elementary Nisnevich squares in Sch_k. The associated topology to this cd-structure is called **Nisnevich topology**.

The small[77] Nisnevich site, unlike the étale, isn't defined solely by its coverings.

Definition 10.2.9 (Small Nisnevich site)**.** For any scheme X, the **small Nisnevich site** X_{Nis} consists of the full category $\acute{\mathrm{E}}\mathrm{t}_X \subset \mathrm{Sch}_{/X}$ equipped with the Nisnevich topology.

Observation 10.2.10. Ex. 10.2.7 shows that the Nisnevich topology is finer than the Zariski topology, since every open immersion is étale. Therefore, the identity on $\mathrm{Sch}_{/k}$ is a morphism on presites, hence yields a canonical geometric morphism by Thm. 2.3.23

$$\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sch}_{/k}) \underset{u_*}{\overset{L_{Nis}}{\leftrightarrows}} \mathrm{Shv}_{\mathrm{Zar}}(\mathrm{Sch}_{/k})$$

Definition 10.2.11. The unstable category of motivic spaces is defined as the two step localization

$$\mathcal{P}(\mathrm{Sm}_{/k}) \xrightarrow{L_{\mathrm{Nis}}} \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_{/k}) \xrightarrow{L_{\mathbb{A}^1_k}} \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_{/k})[S_{\mathbb{A}^1_k} =: \mathcal{S}\mathrm{pc}(k)$$

where $S_{\mathbb{A}^1_k}$ is the collection of morphisms $y(X \times_k \mathbb{A}^1_k) \to y(X)$ induced by the projection for a scheme $X \in \mathrm{Sm}_{/k}$ as in Def. 10.2.1.

The Nisnevich topos admits various different descriptions, listed below without proof.

Lemma 10.2.12 ([MVW06, 12.7])**.** *On* Sch *the Nisnevich topology is equivalently generated by families of étale morphisms* $\{p_i \colon U_i \to X\}_{i \in I}$ *such that for all* $x \in X$ *there is an* $i \in I$ *and an* $u \in U_i$ *and* p_i *induces an isomorphism* $\kappa(u) \cong \kappa(x)$.

---

[77] Note that, similar to the small étale site, the underlying category is not actually small. However, an argument similar to Prop. C.4.10 involving Lem. 2.6.12 can be applied to deduce, that the associated ∞-topos is generated by an actually small site. See also [Hoy14, Appendix C] and [SAG, §3.7].

Lemma 10.2.13 ([BH20, A.2]). *On* $\mathrm{Sch}^{\mathrm{qcqs}}$*, the Nisnevich topology is equivalently generated by the following covering families*

1) *étale maps* $\{U_i \to X\}_{i \in I}$ *which are jointly surjective on all k-points i.e. pulling back along any map* $\mathrm{Spec}(k) \to X$ *induces an surjection* $\coprod_{i \in I} U_i \times \mathrm{Spec}(k) \to \mathrm{Spec}(k)$*.*
2) *étale maps* $\{U_i \to X\}_{i=1,\dots n}$*, such that* $\coprod_{i \in I} U_i \to X$ *admits a finitely presented splitting sequence.*

Lemma 10.2.14 ([MVW06, §12]). *For every henselian local ring A, the global section functor*

$$\Gamma\colon\ \mathrm{Shv}_{\mathrm{Nis}}(A;\mathrm{Set}) \to \mathrm{Set}$$

is an equivalence.

Definition 10.2.15. Let X be a scheme and $Y \in X_{\mathrm{et}}$ étale over X. For any $y \in Y$ consider the geometric morphism

$$(-)_y\colon\ \mathrm{Shv}_{\mathrm{Nis}}(X) \xrightarrow{\times \mathrm{Spec}(\mathcal{O}^h_{Y,y})} \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Spec}(\mathcal{O}_{Y,y})) \xrightarrow{\Gamma} \mathrm{Ani}.$$

Then, for any $F \in \mathrm{Shv}_{\mathrm{Nis}}(X)$, the image F_y of F under this map is called the stalk of F at y.

In order to prove Thm. 10.2.26 we also need the notion of presheaves with transfers. These are defined as objects in $\mathcal{P}(\mathrm{Sm}_{/k})$ which admit a natural interpretation as presheaves on the category of *Correspondences*, which is has the same objects as $\mathrm{Sm}_{/k}$ but whose morphisms are algebraic cycles finite over the source. More precisely, the morphisms are defined as follows.

Definition 10.2.16 ([MVW06, 1.1]). Let k be a field, let $X \in \mathrm{Sm}_{/k}$ be connected and $Y \in \mathrm{Sch}_{/k}$ be separated. An **elementary correspondence** $X \rightsquigarrow^e Y$ is an closed subscheme $W \subset X \times_k Y$, such that $W \to X$ is integral. Thus, $X \rightsquigarrow^e Y$ is pictured as

$$X \twoheadleftarrow W \to Y.$$

If X is not connected, an elementary correspondence $X \rightsquigarrow^e Y$ is an elementary correspondence from a connected component of X to Y.
A **finite correspondence** $X \rightsquigarrow Y$ is a formal linear combination of elementary correspondences

$$\sum_{i=1}^{n} n_i[X \rightsquigarrow^e_i Y n_i \in \mathbb{Z}, n \in \mathbb{N}]$$

Denote by $\mathrm{Cor}_{/\mathrm{k}}(X, Y)$ the group of finite correspondences i.e. the free abelian group of elementary correspondences.

Equivalently, a finite correspondence $X \to Y$ is an algebraic cycle on $X \times_k Y$, which is finite surjective over X.

In order to obtain a category whose morphism spaces are given by correspondences, we need to define compositions of elementary correspondences. This is done by using the Serre intersection product and the push-forward for cycles.

Definition 10.2.17 (Serre intersection product, [Ser00, V.C.1])**.** Let Z_1, Z_2 be closed subvarieties[78] of an $X \in \mathrm{Sm}_{/k}$. Define

$$Z_1 \cdot Z_2 := \sum_{W_j \subset Z_1 \cap Z_2 \text{ irreducible}} n_j [W_j]$$

$$n_j := \sum_{i=0}^{\dim(W_j)} (-1)^i \operatorname{length}_{\mathcal{O}_{X,\eta_j}} \operatorname{Tor}_i^{\mathcal{O}_{X,\eta_j}} \left({\mathcal{O}_{X,\eta_j}}\big/{I_1}, {\mathcal{O}_{X,\eta_j}}\big/{I_2} \right)$$

where η_j is the generic point of W_j in X and $I_1, I_2 \subset \mathcal{O}_{X,\eta_j}$ are the prime-ideals corresponding to Z_1 and Z_2 respectively. Here, $\operatorname{length}_{\mathcal{O}_{X,\eta_j}}$ is the length of an $\mathcal{O}_{X,\eta_j}$-module i.e. the length of the longest chain of submodules.

Definition 10.2.18 (Push-forward, [MVW06, 1.4])**.** Let $p\colon T \to S$ be a morphism of schemes, $W \subset T$ a closed irreducible subset, such that $W_{red} \to S$ is finite. Then, $p(W) =: V$ is a closed irreducible[79] subset of S. Set $d := [\kappa(\eta_W) : \kappa(\eta_V)]$, where η_W, η_V are the corresponding generic points of W resp. V. Define the push-forward of W along p to be the cycle

$$p_* W := d \cdot V.$$

Definition 10.2.19 (Composition of elementary Correspondences,[MVW06])**.** Suppose $V\colon X \rightsquigarrow^e Y$ and $W\colon Y \rightsquigarrow^e Z$ are elementary correspondences. To define their composition, take their intersection product $[T] := (V \times_k Z) \cdot (X \times_k W)$, which is a cycle on $X \times_k Y \times_k Z$ and set $W \circ V := \mathrm{pr}_* T$, which is the push-forward of T along the $\mathrm{pr}\colon X \times_k Y \times_k Z \to X \times_k Z$.

In order to show associativity of the composition, one needs to make sense of compositions of general correspondences. This is done by defining the pullback of a codimension r cycle $Z \subset Y$ along a map $f\colon X \to Y$ of $\mathrm{Sm}_{/k}$ as the cycle $\Gamma_f \cdot (X \times Z)$. Here Γ_f is the graph[80] of f. This is only defined if each component of $f^{-1}(\mathrm{supp}(Z))$ has codimension $\geq r$ (cf. [Ser00, V.C.7], [Ful98, p.18]). Then, for correspondences $V\colon X \rightsquigarrow Y$ and $W\colon Y \rightsquigarrow Z$ one sets

$$\mathrm{pr}_{X \times Z *}(\mathrm{pr}^*_{X \times Y} V \cdot \mathrm{pr}^*_{Y \times Z} W)$$

if the pullbacks are well defined. Here $pr_\bullet$ denote the projections from $X \times Y \times Z$.

Note. Using this general formula and the projection formula of [Ser00, p.118] it is shown in [Ful98, Prop.16.1.1], that compositions of correspondence are associative.

Observation 10.2.20. The identity correspondence $X \rightsquigarrow^e X$ of a separated $X \in \mathrm{Sm}_{/k}$ is simply given by $X \mathrel{-\!\!\!+\!\!\!\to} X \times_k X$. Indeed, if $W\colon X \rightsquigarrow^e Y$ is another correspondence, then $X \times_k Y \cdot X \times_k W \cong X \times W$, since $X \times_k W = X \times_k Y \cap X \times_k W$ is itself irreducible and the higher Tor groups in Def. 10.2.17 vanish.

Definition 10.2.21. Let $\mathrm{Sm}^{\mathrm{sep}}_{/k}$ denote the category of smooth separated schemes of finite

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[78] i.e. Z_1 and Z_2 are integral separated schemes of finite type over k.

[79] Since finite morphisms are integral, in particular universally closed (cf. [Har77]).

[80] i.e. Γ_f is the image of X under $\gamma_f\colon X \to \times_k Y$. Note that this is a closed immersions, since it arises as pullback of the diagonal $Y \mathrel{-\!\!\!+\!\!\!\to} Y \times_k Y$ along (f, id).

type over k.[81] Denote by $\mathrm{Cor}_{/\mathrm{k}}$ the category which has the same objects as $\mathrm{Sm}^{\mathrm{sep}}_{/k}$, but whose morphisms are given by finite correspondences.

Lemma 10.2.22. *There is a fully faithful inclusion* $u\colon \mathrm{Sm}^{\mathrm{sep}}_{/k} \hookrightarrow \mathrm{Cor}_{/\mathrm{k}}$*, which sends a morphism* $f\colon X \to Y$ *to the graph* $\Gamma_F\colon X \mathrel{\rightarrow\!\!\!\!\!\!+}\; X \times_k Y$.

Observation 10.2.23. $\mathrm{Cor}_{/\mathrm{k}}$ is a additive category with zero-object given by and disjoint unions as coproduct.

Definition 10.2.24. A **presheaf with transfers** is an additive functor $F\colon \mathrm{Cor}_{/\mathrm{k}} \to \mathrm{Ab}$. Denote their category by PshT_k. Precomposing with the inclusion u of Lem. 10.2.22 yields a functor $u^*\colon \mathrm{PshT}_k \to \mathcal{P}(\mathrm{Sm}^{\mathrm{sep}}_{/k}; \mathrm{Ab})$. Hence, a presheaf $F \in \mathcal{P}(\mathrm{Sm}^{\mathrm{sep}}_{/k}; \mathrm{Ab})$ is said to admit transfers if it is in the image of u^*. Let $\mathbb{Z}_{\mathrm{tr}}$ denote the composition

$$\mathbb{Z}_{\mathrm{tr}}\colon \mathrm{Sm}^{\mathrm{sep}}_{/k} \xrightarrow{u} \mathrm{Cor}_{/\mathrm{k}} \xrightarrow{y} \mathrm{PshT}_k$$

where y is the Yoneda embedding.

Observation 10.2.25. All constant presheaves in $\mathcal{P}(\mathrm{Sm}^{\mathrm{sep}}_{/k}; \mathrm{Ab})$ admit transfers.

The main Theorem of this section identifies Nisnevich with Zariski cohomology for the previously introduces sheaves.

Theorem 10.2.26 ([MVW06, 13.9])**.** *The forgetful geometric morphism of Obs. 10.2.10* $u_*\colon \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sch}_{/k}) \to \mathrm{Shv}_{\mathrm{Zar}}(\mathrm{Sch}_{/k})$ *induces an isomorphism*

$$H^n_{\mathrm{Nis}}(X, F) \cong H^n_{\mathrm{Zar}}(X, u_*(F)) \ \forall n \leq 0$$

for any smooth separated scheme X over k and $\mathbb{A}^1$-local abelian Nisnevich sheaf F with transfers.

Once Thm. 10.2.26 is proven, one obtains an comparison of shapes, since all constant sheaves naturally are $\mathbb{A}^1$-local and admit transfers.

Corollary 10.2.27. *The restriction of the geometric morphism of Obs. 10.2.10 to*

$$\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}^{\mathrm{sep}}_{/k}) \underset{u_*}{\overset{L_{\mathrm{Nis}}}{\leftrightarrows}} \mathrm{Shv}_{\mathrm{Zar}}(\mathrm{Sm}^{\mathrm{sep}}_{/k})$$

gives rise to an equivalence of shapes via Thm. 4.6.9

$${}^{\mathrm{ab}}\widehat{\Pi}^{\mathrm{Nis}}_{\infty} \xrightarrow{\simeq} {}^{\mathrm{ab}}\widehat{\Pi}^{\mathrm{Zar}}_{\infty}.$$

The following lemma is the first of several needed to prove Theorem 10.2.26.

Lemma 10.2.28 ([MVW06, Ex.12.10])**.** *If* $F \in \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}^{\mathrm{sep}}_{/k}; \mathrm{Ab})$ *admits transfers, then* $H^n_{\mathrm{Nis}}(-, F)$ *is a presheaf with transfers.*

This is an exercise in [MVW06], which is solved by showing, that F admits an Γ-acyclic resolution by sheaves with transfers. It is suggested in [MVW06], to show exactness and the sheaf condition

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[81] The adjective 'finite type over k' is only included for clarity. Separated smooth schemes over k are of finite type.

by hand. However, by relying on[Wei94], this can be avoided as follows.

Proof. Let $X \in \mathrm{Sm}_{/\mathrm{k}}$ be a scheme and $i_X^*\colon \mathrm{Shv}_{\mathrm{Nis}}(Sm_{/k}() \to \mathrm{Shv}_{\mathrm{Nis}}(X)$ the geometric morphism given by taking the slice over X. The Godement resolution ([Wei94, 8.6.15]) of i_X^*F, which is an augmented exact complex in $\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_{/k}; \mathrm{Ab})$ admits transfers, since every entry in this resolution is the direct image of a constant (skyscraper) sheaf. More precisely, the Godement resolution of F is the complex arising from the simplicial resolution given by the triple/monad $\top := x_*x^*$ (cf.[Wei94, 8.6.1]) provided by a geometric morphism
$\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Spec}(K); \mathrm{Ab}) \underset{x_*}{\overset{x^*}{\leftrightarrows}} \mathrm{Shv}_{\mathrm{Nis}}(X; \mathrm{Ab})$ for a closed point $x\colon \mathrm{Spec}(K) \to X$. More more precisely, the simplicial resolution is given by the unit morphisms

$$\begin{tikzcd}
\cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & \prod\limits_{x \in X \text{ closed}} x_*x^*x_*x^*i_X^*F \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \prod\limits_{x \in X \text{ closed}} x_*x^*i_X^*F \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & i_X^*F
\end{tikzcd}$$

By the Dold-Kan correspondence this provides an exact resolution of sheaves on $\mathrm{Sm}_{/k}$. Since all sheaves in $\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Spec}(K); \mathrm{Ab}) \simeq \mathrm{Ab}$ are constant, the skyscraper sheaves participating in this complex admit transfers. ☐

Using the theory of standard triples, outlined in [MVW06, §11] one observes, that the triviality of the Zariski sheafification of $\mathbb{A}^1$-local presheafs with transfers can be checked on points.

Lemma 10.2.29 ([MVW06, 11.2])**.** *Let* $F \in \mathcal{P}(\mathrm{Sm}_{/k}^{\mathrm{sep}}; \mathrm{Ab})$ *be an* $\mathbb{A}^1$*-invariant presheaf with transfers. If* $F(\mathrm{Spec}(K)) = 0$ *for all fields* K/k*, then* $L_{\mathrm{Zar}}F = 0$ *in* $\mathrm{Shv}_{\mathrm{Zar}}(\mathrm{Sm}_{/k}; \mathrm{Ab})$*.*

This is the primary result of [MVW06, §11]. Given the depth of the arguments involved, we omit the proofs of both this and the following theorem, which is one of the main results of the lecture series [MVW06].

Theorem 10.2.30 ([MVW06, 24.1])**.** *Let* k *be a perfect field and let* $F \in \mathcal{P}(\mathrm{Sm}_{/k}^{\mathrm{sep}}; \mathrm{Ab})$ *be an* $\mathbb{A}^1$*-invariant presheaf with transfers. Then,*

$$H^n_{\mathrm{Nis}}(-, L_{\mathrm{Nis}}F)\colon (\mathrm{Sm}_{/k}^{\mathrm{sep}})^{\mathrm{op}} \to \mathrm{Ab}$$

is $\mathbb{A}^1$*-invariant for every* $n \geq 0$*.*

Let $X \in \mathrm{Sm}_{/k}^{\mathrm{sep}}$. By precomposition with the forgetful functor $\mathrm{Sm}_{/X}^{\mathrm{sep}} \subset \mathrm{Sm}_{/k}^{\mathrm{sep}}$, the presheaf $H^n(-, L_{\mathrm{Nis}}F)$ restricts to a presheaf on $\mathrm{Sm}_{/X}^{\mathrm{sep}}$. Recall, that $\mathrm{Shv}_{\mathrm{Nis}}(X) \simeq \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_{/X}^{\mathrm{sep}})$ Let us put this presheaf into context of derived functors.

Observation 10.2.31. Consider the composition

$$X_{Nis}^{\mathrm{op}} \times \mathcal{P}(\mathrm{Sm}_{/X}^{\mathrm{sep}}; \mathrm{Ab}) \xrightarrow{id \times L_{\mathrm{Nis}}} \mathrm{Sm}_{/X}^{\mathrm{sep}} \times \mathrm{Shv}_{\mathrm{Nis}}(X; \mathrm{Ab}) \xrightarrow{\mathrm{Hom}_{\mathrm{Shv}_{\mathrm{Nis}}}(L_{\mathrm{Nis}}y(-), -)} \mathrm{Ab}.$$

Then, by the $\times$-Hom adjunction, this composition is equivalent to

$$\mathcal{P}(\mathrm{Sm}_{/X}^{\mathrm{sep}}; \mathrm{Ab}) \xrightarrow{L_{Nis}} \mathrm{Shv}_{\mathrm{Nis}} \xrightarrow{\gamma} \mathcal{P}(\mathrm{Sm}_{/X}^{\mathrm{sep}}; \mathrm{Ab}), \quad F \mapsto L_{\mathrm{Nis}}F \mapsto [Y \mapsto (L_{\mathrm{Nis}}F(Y))].$$

where γ preserves limits. Since $\mathrm{D}^+(\mathcal{P}(\mathrm{Sm}^{\mathrm{sep}}_{/X};\mathrm{Ab}) \simeq \mathcal{P}(\mathrm{Sm}^{\mathrm{sep}}_{/X};\mathrm{D}^+(\mathrm{Ab})$ and L_{Nis} is exact, this gives rise to the functor

$$\mathcal{P}(\mathrm{Sm}^{\mathrm{sep}}_{/X};\mathrm{D}^+(\mathrm{Ab}) \xrightarrow{L_{\mathrm{Nis}}} \mathrm{D}^+_{\mathrm{Nis}}(X) \xrightarrow{R\,\mathrm{Hom}} \mathcal{P}(\mathrm{Sm}^{\mathrm{sep}}_{/X};\mathrm{D}^+(\mathrm{Ab})), \quad F^\bullet \mapsto R\,\mathrm{Hom}(Ly(-), LF^\bullet)$$

If this is precomposed with $\mathcal{P}(X;\mathrm{Ab}) \to \mathcal{P}(X;\mathrm{D}^+\,\mathrm{Ab}), F \mapsto F[0]$, then the image of F under the resulting functor is $H^\bullet_{\mathrm{Nis}}(-, L_{\mathrm{Nis}}F)$.

Proof of Theorem 10.2.26. Let $u_*\colon \mathrm{Shv}_{\mathrm{Nis}}(X) \to \mathrm{Shv}_{\mathrm{Zar}}(X)$ be the forgetful functor, where both categories are viewed as sheaves over $\mathrm{Sm}^{\mathrm{sep}}_{/X}$. The case $n = 0$ is trivial, since $\Gamma_{\mathrm{Nis}} \simeq \Gamma_{\mathrm{Zar}} \circ u_*$, because Ani is terminal in $\mathcal{RT}\mathrm{op}$.

To verify the other cases observe that u_* can be written as the composition

$$\mathrm{Shv}_{\mathrm{Nis}}(X) \xrightarrow{\gamma} \mathcal{P}(\mathrm{Sm}^{\mathrm{sep}}_{/X};\mathrm{Ab}) \xrightarrow{L_{Zar}} \mathrm{Shv}_{\mathrm{Zar}}(X;\mathrm{Ab}).$$

where γ is the functor of Obs. 10.2.31. Therefore, the following diagram commutes

$$\begin{tikzcd}
\mathrm{D}^+_{\mathrm{Nis}}(X) \arrow[r, "R\,\mathrm{Hom}"] \arrow[rr, bend left, "Ru_*"] \arrow[dr, swap, "R\Gamma_{\mathrm{Nis}}"] & \mathcal{P}(\mathrm{Sm}^{\mathrm{sep}}_{/X}) \arrow[r, "L_{\mathrm{Zar}}"] & \mathrm{D}^+_{\mathrm{Zar}}(X) \arrow[dl, "R\Gamma_{\mathrm{Zar}}"] \\
& \mathrm{D}^+(\mathrm{Ab}) &
\end{tikzcd}$$

Recall from Lem. 10.2.14 that for every henselian ring A, the global section functor
$\Gamma\colon \mathrm{Shv}_{\mathrm{Nis}}(A;\mathrm{Ab}) \to \mathrm{Ab}$ is exact. Therefore, $H^n(A, F) = 0$ for every Nisnevich sheaf F and every henselian local ring A (and in particular every field). Let $i\colon \mathrm{Shv}_{\mathrm{Nis}}(X) \subset \mathcal{P}(\mathrm{Sm}^{\mathrm{sep}}_{/X}$ denote the inclusion. Then, Lem. 10.2.29 implies that

$$L_{\mathrm{Zar}}R\,\mathrm{Hom}(F[0]) = L_{\mathrm{Zar}}H^\bullet(-, F) = L_{\mathrm{Zar}}i(F)[0] = u_*F[0]$$

for all $F \in \mathrm{Shv}_{\mathrm{Nis}}(X;\mathrm{Ab})$, which are $\mathbb{A}^1$-invariant and admit transfers, since in this case each $H^n(-, F)$ is an $\mathbb{A}^1$-invariant presheaf with transfers by Lem. 10.2.28 and Thm. 10.2.30.
Consequently, $Ru_* \simeq L_{\mathrm{Zar}}R\,\mathrm{Hom} \simeq u_*$ is exact and $R\Gamma_{\mathrm{Nis}} \simeq R\Gamma_{\mathrm{Zar}} \circ u_*$. □

10.3 Comparison of Nisnevich and cdh topology

The comparison between the Berkovich and cdh topologies relies fundamentally on the identification of Nisnevich and cdh cohomology for $\mathbb{A}^1$-local abelian sheaves. This identification holds for smooth schemes over fields admitting resolution of singularities. In what follows, we provide an overview of this theory, largely adopting the treatment found in [MVW06].

Definition 10.3.1 (cdh-Topology)**.** A **abstract blow-up square** is a pullback square in Sch_k

$$\begin{tikzcd}
\tilde{Z} \arrow[r] \arrow[d] & \tilde{X} \arrow[d] \\
Z \arrow[r] & X
\end{tikzcd}$$

such that $Z \to X$ is a closed immersion, $\tilde{X} \to X$ is proper and $\tilde{X} \setminus \tilde{Z} \to X \setminus Z$ is an isomorphism. In this case, $Z \twoheadrightarrow X$ is often called the center of the abstract blow-up $\tilde{X} \to X$. The **lower cd-structure** is the class of abstract blow-up squares in Sch_k.

Define the **combined cd-structure** to be the union of the Nisnevich and the lower cd-structure. The **cdh-topology** τ_{cdh} is the topology associated to the combined cd-structure.

Example 10.3.2. As their name suggests every actual blow-up $\tilde{X} \to X$ with center $Z \twoheadrightarrow X$ gives rise to an abstract blow-up square, since $\tilde{X} \to X$ is projective and an isomorphism outside Z (cf. [Har77, II.7.13] for (locally) noetherian scheme or [FOAG, 22.3.2] for the general case).

Example 10.3.3. For any $X \in \mathrm{Sch}_{/k}$ the map $X_{\mathrm{red}} \to X$ is proper and forms a trivial abstract blow-up square with $Z = \emptyset$. Thus, by Lem. 10.2.12, $X_{\mathrm{red}} \to X$ is a proper birational cdh covering.

Example 10.3.4. If $\tilde{X} \to X$ is an abstract blow-up with center Z and Z does not contain a generic point of X, then $\tilde{X} \coprod Z \to X$ is a proper birational cdh covering.

Lemma 10.3.5 ([SV00, 5.8])**.** *A proper morphism $p\colon \tilde{X} \to X$ in Sch/k, where for every $x \in X$, there is an $x' \in p^{-1}(x)$ with $\kappa(x') \cong \kappa(x)$, is a cdh-cover.*

Proof. Let $p\colon \tilde{X} \to X$ be such a proper map. We proceed with an induction over $d := \dim(X)$.

Case: d=0
In this case, $X \cong \mathrm{Spec}(K)$ for a field K/k. By assumption, p admits a section.

Induction Step:
By Ex. 10.3.3 we may assume that X is reduced. Moreover, we can take X to be integral, by considering each connected component individually. By assumption, there exists a lift $\tilde{\eta} \in \tilde{X}$ for every generic point η of (an integral component) of X. Let $X' \twoheadrightarrow \tilde{X}$ be the closure of $\tilde{\eta}$. Then, $q\colon X' \to X$ is a proper birational map. Thus, suppose that q is an isomorphism outside the closed subscheme $Z \subset X'$. Therefore, the proper maps $X' \to X \leftarrow Z$ form an abstract blow-up square thus a cdh-cover. Now, $\tilde{X} \times_X X' \to X'$ is a cdh-cover, as it admits a section and $\tilde{X} \times_X Z \to Z$ is a cdh-cover by the induction hypothesis, as $\dim(Z) < \dim(X)$. In sum, $\tilde{X} \to X$ is cdh-locally a cdh-cover, hence a cdh-cover. ☐

Corollary 10.3.6. *If X is smooth, every blow-up $\tilde{X} \to X$ along a smooth center is a cdh-cover.*

Consequently, over fields admitting resolution of singularities, cdh-sheaves are uniquely determined by their restriction to smooth schemes. However, since $\mathrm{Sm}_{/k}$ does not have enough pullbacks to ensure that all abstract blow-up squares exist, the cdh-topology cannot be defined via a pretopology. Instead, we define the topology by specifying its sieves directly.

Lemma 10.3.7. *Suppose k admits resolution of singularities. Let $u\colon \mathrm{Sm}_{/k} \to \mathrm{Sch}_{/k}$ denote the inclusion. For any $X \in \mathrm{Sm}_{/k}$, let $\mathrm{CovS}_{\mathrm{cdh}}(X)$ denote the collection of cdh-covering sieves on X in $\mathrm{Sch}_{/k}$. The assignment*

$$X \mapsto \mathrm{CovS}^{\mathrm{sm}}_{\mathrm{cdh}}(X) := \{u^{-1}S \mid S \in \mathrm{CovS}_{\mathrm{cdh}}(X)\}$$

defines a Grothendieck topology on $\mathrm{Sm}_{/k}$*, called the* ***cdh-topology*** *on* ***smooth schemes****.*

Proof. We verify the conditions in Def. 2.2.4 of the collection of covering sieves above.

1) For each $X \in \mathrm{Sm}_{/k}$ the slice category $(\mathrm{Sm}_{/k})_{/X} = u^{-1}(\mathrm{Sch}_{/k})$ is a covering sieve.

2) Let $f\colon Y \to X$ be a map in $\mathrm{Sm}_{/k}$ and $S = u^{-1}T \in \mathrm{CovS}^{\mathrm{sm}}_{\mathrm{cdh}}(X)$ a covering sieve. Then,

$$f_!^{-1}S = f_!^{-1}u^{-1}T \cong u^{-1}f_!^{-1}T.$$

since all u is fully faithful (cf. 2.2.3). Therefore, $f_!^{-1}S \in \mathrm{CovS}^{\mathrm{sm}}_{\mathrm{cdh}}(X)$, since $f_!^{-1}T$ is a covering sieve.

3) Let S be some sieve on X in $\mathrm{Sm}_{/k}$ and let $S' \in \mathrm{CovS}^{\mathrm{sm}}_{\mathrm{cdh}}(X)$, say $S' = u^{-1}T'$. Since u is fully faithful, $u(S)$ is a sieve on X in $Sch_{/k}$. Suppose that for every $f\colon Y \to X$ in S', the sieve $f_!^{-1}S$ is in $\mathrm{CovS}^{\mathrm{sm}}_{\mathrm{cdh}}(Y)$. This implies that each sieve $uf_!^{-1}S = f_!^{-1}u(S)$ is a covering sieve on Y in $\mathrm{Sch}_{/k}$ and therefore, $u(S)$ is a covering sieve on X in $\mathrm{Sch}_{/k}$. However, $u^{-1}u(S) = S$ and therefore S is a covering sieve on X in $\mathrm{Sm}_{/k}$. ☐

Lemma 10.3.8 ([MVW06, p.97]). *If k admits resolutions of singularities, every cdh-covering $\{Y_i \to X\}_{i\in I}$ in $\mathrm{Sch}_{/k}$ admits a refinement $\{\tilde{Y}_j \to X\}_{j\in J}$, where X is cdh-covered by $\tilde{Y}_j \in \mathrm{Sm}_{/k}$.*

Proposition 10.3.9 ([SV00, 5.11]). *Suppose k admits resolution of singularities. The inclusion $u\colon \mathrm{Sm}_{/k} \hookrightarrow \mathrm{Sch}_{/k}$ induces an equivalence*

$$\mathrm{Shv}_{\mathrm{cdh}}(\mathrm{Sm}_{/k}) \simeq \mathrm{Shv}_{\mathrm{cdh}}(\mathrm{Sch}_{/k}).$$

The proof of [SV00, 5.11] directly generalizes to anima valued sheaves.

Proof. The geometric morphism $g_*\colon \mathcal{P}(\mathrm{Sch}_{/k}) \to \mathcal{P}(\mathrm{Sm}_{/k})$ obtained by precomposition with u sends cdh-sheaves to cdh-sheaves, since for every cdh-covering $p\colon Y \to X$ there is a cdh-covering $\tilde{p}\colon \tilde{Y} \to X$ with $\tilde{Y} \in \mathrm{Sm}_{/k}$ refining p and thus

$$g_*F(X) \simeq F(X) \simeq \lim\nolimits_{y\in S_p} F(y) \simeq \lim\nolimits_{\tilde{y}\in S_{\tilde{p}}} F(\tilde{y}) \simeq \lim\nolimits_{\tilde{y}\in S_{\tilde{p}}} g_*F(\tilde{y})$$

where S_p and $S_{\tilde{p}}$ are sieves generated by p resp. $\tilde{p}$. The last equivalence persists, because $S_{\tilde{p}}$ is generated by smooth schemes (cf. Prop. 2.3.19). Let $\iota\colon \mathrm{Shv}_{\mathrm{cdh}} \hookrightarrow \mathcal{P}(\mathrm{Sm}_{/k})$ denote the inclusion Let $r := \mathrm{Ran}_y u\colon \mathcal{P}(\mathrm{Sm}_{/k}) \to \mathcal{P}(\mathrm{Sch}_{/k})$ denote the right Kan extension of u along the Yoneda embedding. Then, r preserves cdh-sheaves, as for any $X \in \mathrm{Sch}_{/k}$ and any cdh-covering $p\colon U \to X$, there is a cdh-covering $\tilde{p}\colon \tilde{U} \to X$ refining $p\colon U \to X$, which is generated by smooth schemes and for the associated covering sieves S_p,$S_{\tilde{p}}$ in $\mathrm{Sch}_{/k}$ and for any $F \in \mathrm{Shv}_{\mathrm{cdh}}(\mathrm{Sm}_{/k})$, we have:

$$\begin{aligned}\lim_{V\in S_X} r\iota F(V) &\simeq \lim_{V\in S_{\tilde p}} r\iota F(V) \simeq \lim_{V\in S_{\tilde p}} \mathrm{map}_{\mathcal{P}(\mathrm{Sch}_{/k})}(y(V), r\iota F)\\ &\simeq \lim_{V\in S_{\tilde p}} \mathrm{map}_{\mathrm{Shv}(\mathrm{Sm}_{/k})}(L_{\mathrm{cdh}} g_* y(V), F)\\ &\simeq \lim_{V\in S_{\tilde p}} \mathrm{map}_{\mathrm{Shv}(\mathrm{Sm}_{/k})}(g_* L_{\mathrm{cdh}} y(V), F)\\ &\simeq \mathrm{map}_{\mathrm{Shv}(\mathrm{Sm}_{/k})}(\operatorname*{colim}_{V\in S_{\tilde p}} g_* L_{\mathrm{cdh}} y(V), \iota F)\end{aligned}$$

However, since every $V \in S_{\tilde p}$ admits a cdh cover by smooth schemes, every $L_{\mathrm{cdh}} y(V)$ for $V \in S_{\tilde p}$ is colimit of $L_{\mathrm{cdh}} y(V')$ for $V' \in u^{-1} S_{\tilde p}$. Hence,

$$\begin{aligned}&\mathrm{map}_{\mathrm{Shv}(\mathrm{Sm}_{/k})}(\operatorname*{colim}_{V\in S_{\tilde p}} g_* L_{\mathrm{cdh}} y(V), F) \simeq \mathrm{map}_{\mathrm{Shv}(\mathrm{Sm}_{/k})}(\operatorname*{colim}_{V\in u^{-1}S_{\tilde p}} L_{\mathrm{cdh}} y(V), F)\\ &\simeq \mathrm{map}_{\mathrm{Shv}(\mathrm{Sm}_{/k})}(L_{\mathrm{cdh}} g_* y(X), F) \simeq \mathrm{map}_{\mathcal{P}(\mathrm{Sch}_{/k})}(y(X), r\iota F)\\ &\simeq r\iota F(X)\end{aligned}$$

Moreover, r and g_* are mutual inverses, since the unit induces equivalences for every $F \in \mathcal{P}(\mathrm{Sch}_{/k})$ and $X \in \mathrm{Sch}_{/k}$:

$$r g_* F(X) \simeq \mathrm{map}_{\mathcal{P}(\mathrm{Sm}_{/k})}(g_* y(X), g_* F) \simeq \mathrm{map}_{\mathcal{P}(\mathrm{Sch}_{/k})}(y(X), F) \simeq F(X)$$

because g_* inherits fully faithfulness from u. Likewise the counit induces an equivalence for every $G \in \mathcal{P}(\mathrm{Sm}_{/k})$ and $Y \in \mathrm{Sm}_{/k}$

$$\begin{aligned}g_* r G(Y) &\simeq rG(u(Y)) \simeq \mathrm{map}_{\mathcal{P}(\mathrm{Sch}_{/k})}(y(u(Y)), rG)\\ &\simeq \mathrm{map}_{\mathcal{P}(\mathrm{Sm}_{/k})}(g_* y(u(Y)), G) \simeq \mathrm{map}_{\mathcal{P}(\mathrm{Sm}_{/k})}(y(Y), G) \simeq G(Y).\end{aligned} \qquad \square$$

As previously announced, the main objective of this subsection is to establish the comparison below.

Theorem 10.3.10. *If k admits resolutions of singularities, then L_{cdh} induces an isomorphism*

$$H^n_{\mathrm{Nis}}(X, F) \cong H^n_{\mathrm{cdh}}(X, L_{\mathrm{cdh}}(F)) \ \forall n \leq 0$$

for any smooth scheme X over k and any $\mathbb{A}^1$-local abelian Nisnevich sheaf F.

Convention 10.3.11. For the remainder of this subsection, assume that k admits resolutions of singularities.

In this case, Lem. 10.3.8 yields the following Corollary.

Corollary 10.3.12 ([MVW06, 12.30])**.** *An $F \in \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_{/k}; \mathrm{Ab})$ has $L_{\mathrm{cdh}} F = 0$ if and only if any $X \in \mathrm{Sm}_{/k}$ and any $a \in F(X)$ there is a finite sequence of blow-ups $p\colon X_r \to \cdots \to X_1 \to X$ along smooth centers such that $p^*(a) = 0 \in F(X_r)$.*

Lemma 10.3.13. *Let $h\colon X' \to X$ be a cdh cover of $X \in \mathrm{Sm}_k$ and let $C \in \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k; \mathrm{Ab})$ be* *the sheaf cokernel of the induced map* $h_* := y_{\mathrm{ab}}(h)\colon y_{\mathrm{ab}}(X') \to y_{\mathrm{ab}}(X)$ *in* $\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k; \mathrm{Ab})$*, then* $L_{\mathrm{cdh}}(C) = 0$

Proof. Pick some $U \in \mathrm{Sm}_k$. Then, $C(U) \cong \mathrm{Hom}(U, X)/h_*(\mathrm{Hom}(U, X'))$ since colimits are calculated pointwise. Pick a generator $f \in \mathrm{Hom}(U, X)$ for $C(U)$. Consider the pullback of h along f

$$\begin{tikzcd}
U' \arrow[r, "g"] \arrow[d, "f'"] & U \arrow[d, "f"] \\
X' \arrow[r, "h"] & X
\end{tikzcd}$$

Precomposition with the cdh-cover g induces a map $g^*\colon C(U) \to C(U')$. Since $g_*(f) = h_*(f')$ for $f' \in \mathrm{Hom}(U', X')$, the generator f maps to 0 in $C(U')$. However, $L_{\mathrm{cdh}}C(U) \cong L_{\mathrm{cdh}}C(U')$ since g is a cdh-cover, so $L_{\mathrm{cdh}}(f) = 0 \in L_{\mathrm{cdh}}(C(U))$. As this argument applies to every $f \in C(U)$ (with possibly different g and U') one concludes that $L_{\mathrm{cdh}}C = 0$. ☐

Lemma 10.3.14 ([SV00, 5.12])**.** *Suppose k admits resolutions of singularities. Let* $H \in \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}^{\mathrm{sep}}_{/k}; \mathrm{Ab})$*, such that* $L_{\mathrm{cdh}}(H) = 0$*. Let* $F \in \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}^{\mathrm{sep}}_{/k})$ *be a* $\mathbb{A}^1$*-local Nisnevich sheaf with transfers. Then,*

$$\mathrm{Ext}^n_{\mathrm{Nis}}(H, F) = 0, \qquad \forall n \geq 0.$$

Proof. As usual, it suffices to consider the restrictions of the sheaves in question to an arbitrary $X \in \mathrm{Sm}^{\mathrm{sep}}_{/k}$. First, consider the canonical surjection

$$q\colon \oplus_{a \in H(X)}\, y^{\mathrm{ab}}(X) \to H$$

By Cor 10.3.12, each $a \in H(X)$ vanishes when restricted to a specific cdh-cover $p_a\colon X'_a \to X$. Therefore, the composition

$$a \circ y^{\mathrm{ab}}(p_a)\colon y^{\mathrm{ab}}(X'_a) \to y^{\mathrm{ab}}(X) \to H$$

is zero for the associated map $a \in \mathrm{Hom}(y^{\mathrm{ab}(X),H} \cong H(X)$. Thus, q factors as

$$\oplus_{a \in H(X)} y^{\mathrm{ab}})(X) \to \oplus_{a \in H(X)} \mathrm{coker}(y^{\mathrm{ab}}(p_a)) \xrightarrow{c} H.$$

Moreover, $\mathrm{Ext}^{\bullet}(\mathrm{coker}(y^{\mathrm{ab}}(p_a), F) = 0$ by [MVW06, 13.21]. Thus, applying $\mathrm{Ext}(\text{-}, F)$ to the short exact sequence

$$0 \longrightarrow \ker(c) \longrightarrow \oplus_{a \in H(X)} \mathrm{coker}(p_a) \xrightarrow{c} H \longrightarrow 0$$

yields a long exact sequence, which implies $\mathrm{Ext}^0(H, F) = 0$ and $\mathrm{Ext}^n(H, F) \cong \mathrm{Ext}^{n-1}(\ker(c), F)$. Proceeding by an induction on n, with H being a free variable, we notice that the induction start $n = 0$ has already been proved. Now, $L_{\mathrm{cdh}} \ker(c) = 0$, since $L_{\mathrm{cdh}} \mathrm{coker}(p_a) = 0$ for all $a \in H(X)$ by Lem. 10.3.13. Thus, we may apply the induction hypothesis to $\ker(c)$ and conclude

$$0 \cong \mathrm{Ext}^{n-1}(\ker(c), F) \cong \mathrm{Ext}^n(H, F).$$ ☐

Proof of Thm. 10.3.10. Let $u_*\colon \mathrm{Shv}_{\mathrm{cdh}}(\mathrm{Sm}^{\mathrm{sep}}_{/k}) \to \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}^{\mathrm{sep}}_{/k})$ be the forgetful functor i.e. $L_{\mathrm{cd}} \dashv u_*$. Throughout this proof fix an arbitrary $\mathbb{A}^1$-invariant $F \in \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}^{\mathrm{sep}}_{/k}; \mathrm{Ab})$ with with transfers. First, consider the case $n = 0$ i.e.

1) *Claim: F is a cdh-sheaf.*
Let $X' \to X$ be a cdh-cover of X. Let $C := \mathrm{coker}(y^{\mathrm{ab}}(X') \to y^{\mathrm{ab}}(X)) \in \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}^{\mathrm{sep}}_{/k}; \mathrm{Ab})$ be the sheaf cokernel. By left-exactness

$$0 \to \mathrm{Hom}(C, F) \to \mathrm{Hom}(y^{\mathrm{ab}}(X), F) \to \mathrm{Hom}(y^{\mathrm{ab}}(X'), F)$$

is exact. By Lem. 10.3.13, $L_{\mathrm{cdh}} C = 0$ and thus $\mathrm{Hom}(C, F) = 0$ by Lem. 10.3.14. Therefore the sequence above implies that $F(X) \to F(X') \cong L_{\mathrm{cdh}} F(X)$ is an injection for every $X \in \mathrm{Sm}^{\mathrm{sep}}_{/k}$. Consider canonical short exact sequence of this injection of Nisnevich sheaves

$$0 \longrightarrow F \xrightarrow{i} u_* L_{\mathrm{cdh}} F \longrightarrow \mathrm{coker}(i) \longrightarrow 0$$

By Lem. 10.3.13, $L_{\mathrm{cdh}}\, \mathrm{coker}(i) = 0$ and thus $\mathrm{Ext}^1(F, \mathrm{coker}(i)) = 0$. Therefore, this sequence splits (cf. [Wei94, 3.4.1]) and this exhibits F as direct summand of a cdh-sheaf. Thus, F is a cdh-sheaf itself.

2) *Claim*: u_* preserves injective resolutions of F.
Let $0 \to F \to I^\bullet$ be an injective resolution of abelian cdh-sheaves. It suffices to show that $0 \to F \to u_* I^\bullet$ is exact, as u_* preserves injective objects, since L_{cdh} is exact (cf.[Sta25, 015Z]). Considering the cohomology sheaves $H^n := H^n(u_* I^\bullet)$, this is equivalent to showing that $H^n = 0$ for all n. This will be proved by induction on n.

Case n=0:
The exactness of $F \hookrightarrow u_* I^0 \to u_* I_1$ follows from the fact that u_* is left exact, as it is right adjoint to L_{cdh}. **Induction Step:**
Consider the exact sequence

$$0 \to F \to u_* I^0 \xrightarrow{i_0} \cdots \xrightarrow{i_{n-2}} u_* I^{n-1} \xrightarrow{i_{n-1}} u_* I^n \xrightarrow{i_n} \mathrm{im}(i_n) \to 0.$$

Since $L_{\mathrm{cdh}} H^n = 0$, Lem. 10.3.14 implies that $\mathrm{Ext}^{n+1}(H^n, F) \cong 0$. Considering the long exact sequences of Ext associated to $0 \to \ker(i_{j-1} \to u_* I^j \to \mathrm{im}(i_j) \to 0$ for each $0 < j < n$ and using that each $u_* I^j$ is injective, provides the following isomorphisms (cf.[Wei94, Ex.2.4.3])

$$\begin{aligned} 0 &\cong \mathrm{Ext}^{n+1}(H^n, F) = \mathrm{Ext}^{n+1}(H^n, \ker(i_0)) \cong \mathrm{Ext}^n(H^n, \mathrm{im}(i_1)) \\ &= \mathrm{Ext}^n(H^n, \ker(i_2)) \cong \mathrm{Ext}^{n-1}(H^n, \mathrm{im}(i_2)) \cong \cdots \cong \mathrm{Ext}^1(H^n, \mathrm{im}(i_n)). \end{aligned}$$

Therefore, $0 \to \mathrm{im}(i_n) \to \ker(i_{n+1}) \to H^n \to 0$ splits and this exhibits H^n as direct summand of a subsheaf of the cdh-sheaf $u_* I^{n+1}$. Thus, H^n is cdh-sheaf itself and therefore $H^n \cong L_{\mathrm{cdh}} H^n \cong 0$.

In sum, picking an injective resolution $F \to I^\bullet$ of cdh-sheaves, we get

$$H^\bullet_{\mathrm{cdh}}(X, L_{\mathrm{cdh}} F) = H(I^\bullet(X)) \cong H(u_* I^\bullet(X)) = H^\bullet_{\mathrm{Nis}}(X, F) \qquad \forall X \in \mathrm{Sm}^{\mathrm{sep}}_{/k}. \qquad \square$$

10.4 Comparison of cdh and analytic Topology

To establish an comparison a comparison between cdh-sheaves on arbitrary schemes locally of finite type and analytic spaces equipped with the open set topology, it is convenient to establish general properties of well-behaved cd structures first.

Definition 10.4.1. A cd-structure $\mathcal{Q}$ on $\mathcal{C}$ is **complete** if and only if it satisfies the conditions

1) If a morphism $c \to \emptyset_{\mathcal{C}}$ appears in a $\mathcal{Q}$-distinguished square, then $c \in \mathcal{C}$ is an initial object.

2) $\mathcal{Q}$ is stable under base-change along arbitrary morphisms, i.e. for any morphism $y \to x$ in $\mathcal{C}$ and any $\mathcal{Q}$-distinguished square

$$\begin{tikzcd} w \arrow[r] \arrow[d] & v \arrow[d] \\ u \arrow[r] & x \end{tikzcd}$$

the following square exists and is also contained in $\mathcal{Q}$

$$\begin{tikzcd} w \times_x y \arrow[r] \arrow[d] & v \times_x y \arrow[d] \\ v \times_x y \arrow[r] & y \end{tikzcd}$$

Definition 10.4.2. A cd-structure $\mathcal{Q}$ on $\mathcal{C}$ is **regular** if and only if it satisfies the conditions

1) Every $\mathcal{Q}$-square is a pullback square.

2) $\mathcal{Q}$ is stable under base-change along arbitrary morphisms.

3) For a $\mathcal{Q}$-square $\begin{tikzcd} w \arrow[r] \arrow[d] & v \arrow[d] \\ u \arrow[r] & x \end{tikzcd}$ at least one of the maps $u \to x$ and $v \to x$ is a monomorphism.

4) For any $\mathcal{Q}$-square as above the following square is also in $\mathcal{Q}$.

$$\begin{tikzcd} w \arrow[r] \arrow[d, "\Delta"] & v \arrow[d, "\Delta"] \\ w \times_u w \arrow[r] & v \times_x v \end{tikzcd}$$

Observation 10.4.3. If condition *1)* is satisfied, then the square in condition *4)* is a pullback square. To see this, apply the pullback pasting law to the left diagram below, to observe that the outer square of that diagram forms a pullback.

$$\begin{tikzcd} w \times_u w \arrow[r] \arrow[d] & w \arrow[r] \arrow[d] & u \arrow[d] \\ v \times_x v \arrow[r] & v \arrow[r] & x \end{tikzcd} \qquad \begin{tikzcd} w \arrow[r] \arrow[d] & w \times_u w \arrow[r] \arrow[d] & u \arrow[d] \\ v \arrow[r] & v \times_x v \arrow[r] & x \end{tikzcd}$$

Since the right inner square of the right diagram above is a pullback, the pasting law implies, that the left inner square of the right diagram is a pullback as well.

Lemma 10.4.4. *The Nisnevich cd-structure is complete.*

Proof. An open subset of an $\emptyset$ is empty and likewise an étale map with codomain $\emptyset$ has empty domain. Moreover, open immersions and étale morphisms are stable under pullback. ☐

Lemma 10.4.5. *The lower cd-structure is complete and regular.*

Proof. Closed immersions and proper morphisms are stable under pullback [GW20, Prop.4.32]. Therefore, pulling back an abstract blow up square induces an isomorphisms of the reduced induced structures. Hence, the lower cd-structure is complete.

The conditions *1)-3)* for regularity are trivially satisfied. Moreover, for any abstract blow up square, all morphisms appearing in the associated square in condition *4)* are closed immersions or proper morphisms, as proper maps and closed immersions are separated and stable under pullback. One readily checks that the diagonals behave well regarding complements, which exhibits the square of condition *4)* as abstract blow-up square as well. ☐

Definition 10.4.6. Let $\mathcal{Q}$ be a cd-structure on $\mathcal{C}$. A presheaf $F \in \mathcal{P}(\mathcal{C})$ satisfies **$\mathcal{Q}$-excision** if F sends initial objects of $\mathcal{C}$ to terminal objects and $\mathcal{Q}$-distinguished squares to pullback squares.

Proposition 10.4.7 ([Voe08, 2.9, 2.15], [AHW17, 3.2.5])**.** *Let $\mathcal{Q}$ be a cd-structure on $\mathcal{C}$.*

1) *If $\mathcal{Q}$ is complete and $F \in \mathcal{P}(\mathcal{C})$ satisfies $\mathcal{Q}$-excision, then F is a $\tau_{\mathcal{Q}}$-sheaf.*
2) *If $\mathcal{Q}$ is complete and regular, then $F \in \mathcal{P}(\mathcal{C})$ satisfies $\mathcal{Q}$-excision if and only if it is a $\tau_{\mathcal{Q}}$-sheaf.*

The proof of *2)* is motivated by [AHW17, 3.2.5].

Proof. ***1)*** **Claim:** *F is local with respect to all covering sieves of initial objects $\emptyset_{\mathcal{C}}$*
By assumption $1_{\mathcal{D}} := F(\emptyset_{\mathcal{C}})$ is a terminal object. For the empty covering sieve $\mathcal{C}^{\emptyset}$ of $\emptyset_{\mathcal{C}}$ we have $F(\emptyset_{\mathcal{C}}) = *_{\mathcal{D}} \simeq \lim_{\mathcal{C}^{\emptyset}}$. Any non-empty covering sieve S of $\emptyset_{\mathcal{C}} \in \mathcal{C}$ is generated by a $\mathcal{Q}$-square, which consists solely of initial objects by completeness. Such a covering sieve is contractible, as the collection of initial objects is contractible. Thus, $\lim_{Y \in S} \simeq F(Y)$ for some $Y \in S$. Since Y is an initial object, this yields $F(Y) \simeq 1_{\mathcal{D}} = F(\emptyset_{\mathcal{C}})$.

Claim: *F is local with respect to all covering sieves of non-initial objects.*
Let $x \in \mathcal{C}$ be not an initial object. Let S_x be a covering sieve of x generated by maps $v \to x$ and $w \to x$, which appear in a $\mathcal{Q}$-square

$$\begin{tikzcd}
w \arrow[r] \arrow[d] & v \arrow[d] \\
u \arrow[r] & x
\end{tikzcd}$$

Abbreviate $u^i := u^{\times_x^i}$ and $v^j := v^{\times_x^j}$ for all $i, j \geq 0$. By the ∞-sheaf condition Prop. 2.3.19, the presheaf F is a $\tau_{\mathcal{Q}}$-sheaf if

$$F(x) \simeq \lim_{[n] \in \mathrm{N}(\Delta^{\mathrm{op}})} F(\check{C}(v \coprod u \to x)_n)$$
$$\overset{2.3.19}{\simeq} \lim \left(F(u) \times F(v) \rightrightarrows F(u \times_x u) \times F(w) \times F(v \times_x v) \overset{\rightrightarrows}{\to} F(u^3) \times F(u \times_x w) \times F(v \times_x w) \times F(v^3) \cdots \right))$$

By Def. 10.4.1.2), all squares of the form

$$\begin{tikzcd}
w \times_x u^i \times_x v^j \arrow[r] \arrow[d] & u^i \times_x v^{j+1} \arrow[d] \\
u^{i+1} \times_x v^j \arrow[r] & u^i \times_x v^j
\end{tikzcd}$$

are $\mathcal{Q}$-distinguished squares for all $i, j \geq 0$. As all these squares are send to pullback squares under F, the sheaf condition of Prop. 2.3.19 is satisfied.

2) By *1)* $\mathcal{Q}$-excision implies that F is local with respect to all covering sieves of initial objects $\emptyset_{\mathcal{C}}$. Conversely, if F is local with respect to the empty sieve $S_\emptyset$, which by completeness is a covering sieves of $\emptyset_{\mathcal{C}}$, then $F(\emptyset_{\mathcal{C}}) \simeq \lim_{S_\emptyset}$ is a terminal object.

To prove the general case[82] consider a $\mathcal{Q}$-distinguished square Q of the form

$$\begin{tikzcd}
w \arrow[r] \arrow[d] & v \arrow[d] \\
u \arrow[r] & x
\end{tikzcd}$$

For any such square, let $K_Q := y(u) \coprod_{y(w)} y(v)$ be the associated pushout and $k_Q : K_Q \to y(x)$ the canonical morphism. The task is to show that F is a $\tau_{\mathcal{Q}}$-sheaf if and only if k_Q induces an equivalence after applying $\mathrm{map}_{\mathcal{P}(\mathcal{C})}(-, F)$ for every Q. To do so, abbreviate $\check{c}_Q := \underset{\mathrm{N}(\Delta^{\mathrm{op}})}{\mathrm{colim}}\, \check{\mathrm{C}}(y(u) \coprod y(v) \to y(x))$ and consider the pullback square

$$\begin{tikzcd}
\check{c}_Q \times_{y(x)} K_Q \arrow[r, "p_1"] \arrow[d, "p_2"] & K_Q \arrow[d, "k_Q"] \\
\check{c}_Q \arrow[r] & y(x)
\end{tikzcd}$$

By Prop. 2.3.19, $\check{c}_Q \to y(x)$ becomes an equivalence after applying $\mathrm{map}_{\mathcal{P}(\mathcal{C})}(-, F)$ for every Q if and only F is a $\tau_{\mathcal{Q}}$-sheaf. Likewise, F satisfies $\mathcal{Q}$-excision if and only if $\mathrm{map}(k_Q, F)$ is an equivalence for every $\mathcal{Q}$-square Q. Thus, it suffices to show that both p_1 and p_2 become equivalences after applying $\mathrm{map}_{\mathcal{P}(\mathcal{C})}(-, F)$. If this is the case, the two conditions are equivalent via the by 2-out-of-3 property of equivalences. For the remaining proof we omit all occurrences of the Yoneda embedding y.

Let us start with p_2. The assertion is trivial, if F satisfies $\mathcal{Q}$-excision. Suppose F is a $\mathcal{Q}$-sheaf. Let $y \in \{u, v, w\}$. Since colimits are universal, we have

$$\check{c}_Q \times_x y = (\underset{\mathrm{N}(\Delta^{\mathrm{op}})}{\mathrm{colim}}\, \check{\mathrm{C}}(u \coprod v) \to x) \times_x y \simeq \underset{\mathrm{N}(\Delta^{\mathrm{op}})}{\mathrm{colim}}\, \check{\mathrm{C}}(u \times_x y \coprod v \times_x y \to y)$$

But $\underset{\mathrm{N}(\Delta^{\mathrm{op}})}{\lim}\, \mathrm{map}_{\mathcal{P}(\mathcal{C})}((\check{\mathrm{C}}(u \times_x y \coprod v \times_x y \to y)), F) \to F(y)$ is an equivalence, since $Q \times_x y$ is a $\mathcal{Q}$-distinguished square and F is a $\tau_{\mathcal{Q}}$-sheaf by assumption. Therefore, $\mathrm{map}(p_2, F) : \mathrm{map}(\check{c}_Q \times_x K_Q, F) \to \mathrm{map}(K_Q, F)$ is an equivalence.

p_1 trivially induces an equivalence after applying $\mathrm{map}(-, F)$ if F is a $\tau_{\mathcal{Q}}$-sheaf. In case F satisfies $\mathcal{Q}$-excision the argument is a bit more complicated. Let $Q_{i,j}$ denote the square obtained from

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[82] Here we provide another argument, why a presheaf which satisfies $\mathcal{Q}$-excision is a $\tau_{\mathcal{Q}}$-sheaf, in case the reader is not convinced by the previous argument.

Q by applying $- \times_x u^i \times_x v^j$, where $i, j \geq 0, i+j \geq 1$. Any such square $Q_{i,j}$ is a $\mathcal{Q}$-square by completeness. According to these conventions, $K_{Q_{i,j}}$ denotes the corresponding pushout of $Q_{i,j}$ and $k_{Q_{i,j}} : K_{Q_{i,j}} \to u^i \times_x v^j$ the canonical map. In consequence,

$$p_1 \simeq \lim_{[n] \in \mathrm{N}(\Delta^{\mathrm{op}})} \coprod_{i+j=n+1} k_{Q_{i,j}}$$

First, assume, that by some lucky coincident *both* arrows $u \to x$ and $v \to x$ appearing in Q are monomorphisms. Then, at least one leg of $Q_{i,j}$, that is $u^{i+1} \times_x v^j \to u^i \times_x v^j$ or $u^i \times_x v^{j+1} \to u^i \times_x v^j$, is an equivalence by 2.6.7, even if $i = 0$ or $j = 0$. Therefore, $\mathrm{map}(k_{Q_{i,j}}, F)$ is an equivalence.

Let us argue for the general case of an arbitrary $\mathcal{Q}$-square Q now. By regularity we may assume that the lower horizontal $u \to x$ of the square Q is a monomorphism. The same argument as above implies, that $\mathrm{map}(k_{Q_{i,j}}, F)$ is an equivalence, once $i \geq 1$. Therefore, it suffices to prove, that $k_{Q_{0,j}}$ induces an equivalence for any $j \geq 1$. Consider the diagram

$$\begin{tikzcd}
v^j \times_x u \arrow[r] \arrow[d, "id_{v(j-1)} \times_x \Delta_v \times_x id_u"'] & v^j \arrow[d, "id_{v(j-1)} \times_x \Delta_v"] \\
v^j \times_x w \arrow[r] \arrow[d] & v^{j+1} \arrow[d] \\
v^j \times_x u \arrow[r] & v^j
\end{tikzcd}$$

where the lower square is given by $Q_{0,j}$. The upper square Q' of this diagram is a $\mathcal{Q}$-square by condition *2)* and 4) of Def. 10.4.2. Since its legs are monomorphisms, this yields $\mathrm{map}(K_{Q'}, F) \simeq F(v^{j+1})$. By definition of the diagonal Δ_v, the vertical compositions are equivalent to the identity, which shows, that

$$\begin{aligned}
F(v^j) &\simeq F(v^j \times_x u) \times_{F(v^j \times_x w)} F(v^j \times_x w) \times_{F(v^j \times_x u)} F(v^j) \\
&\simeq F(v^j \times_x u) \times_{F(v^j \times_x w)} \mathrm{map}(K_{Q'}, F) \\
&\simeq F(v^j \times_x u) \times_{F(v^j \times_x w)} F(v^{j+1}) \\
&\simeq \mathrm{map}(K_{Q_{0,j}}, F). \qquad \square
\end{aligned}$$

The strategy presented in [MV24] to establish a comparison between locally finite type schemes in the cdh topology and their analytifications mainly consists in showing that the the presheaf given by the singular cochain complex is satisfies cdh-excision in the sense of Def 10.4.6.

Notation 10.4.8. Let $|\text{-}| \colon \mathrm{Ber}_k \to \mathrm{Top}$ denote the forgetful functor, which sends a k-analytic space X to its underlying topological space.
Given a ring R and an R-module M, let $C^\bullet_M$ denote the presheaf

$$(\mathrm{Sch}^{\mathrm{lft}}_{/k})^{\mathrm{op}} \to \mathrm{Ch}^+(R),\ X \mapsto C^\bullet_{\mathrm{sing}}(|X^{\mathrm{an}}|, M).$$

where $C^\bullet_{\mathrm{sing}}$ is the cochain complex of singular cohomology.

When restricted to schemes, which are locally embeddable in smooth schemes, assigning a scheme X the sheaf cohomology complex $R\Gamma_{\mathrm{open}}(|X^{\mathrm{an}}|, \underline{M})$ gives an equivalent functor, since $|X^{an}|$ is locally contractible and thus the cohomology theories coincide (cf.[Sel16]).

Motivated by the classical paper [DI04], Vezzani [Vez19] showed, that $|\text{-}|\colon \mathrm{Ber}_k \to \mathrm{Top}$ gives rise to a left Quillen functor, providing the following special case.

Lemma 10.4.9 ([Vez19, 2.13, 2.15])**.** *The functor* $|\text{-}|\colon \mathrm{Ber}_k \to \mathrm{Top}$ *sends elementary Nisnevich squares to homotopy pushout squares.*

Proof. Observe that $|\text{-}|$ preserves pullbacks. Since étale morphisms in Ber_k are open (Lem. 6.2.2), the composition of morphisms of presites

$$u\colon (\mathrm{Ber}_k, \tau_{\mathrm{Nis}}) \subset (\mathrm{Ber}_k, \acute{e}t) \xrightarrow{|\text{-}|} (\mathrm{Top}, \tau_{\mathrm{open}})$$

induces a geometric morphism $u^*\colon \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Ber}_k) \to \mathrm{Shv}_{\mathrm{open}}(\mathrm{Top})$ via Thm. 2.3.23. Fix an arbitrary elementary Nisnevich square in Ber_k

$$\begin{tikzcd} W \arrow[r] \arrow[d] & V \arrow[d] \\ U \arrow[r] & X \end{tikzcd}$$

then,

$$\mathrm{colim}_{[n]\in \mathrm{N}(\Delta^{\mathrm{op}})} \check{\mathrm{C}}(y(|U|)\coprod y(|V|) \to y(|X|))_n \simeq y(|X|)$$

since u^* preserves effective epimorphisms and groupoids are effective. Equivalently,

$$|X| \cong \mathrm{hocolim}_{[n]\in \mathrm{N}(\Delta^{\mathrm{op}})}(|U|\coprod|V| \to |X|).$$

Set $Y := |U|\coprod|V|$. By [DI04, A.7] this homotopy colimit is equivalent to the following actual colimit in Top

$$|X| \simeq \mathrm{coeq}(Y \times_X Y \rightrightarrows Y)$$

By [Ber99b, 5.11] the right hand side is isomorphic to the quotient space

$$|U|\coprod|V| \Big/ (|U| \times_{|X|} |U|)\coprod W \coprod(|V| \times_{|X|} |V|) \cong |U|\coprod|V| \Big/ W$$

which is precisely the pushout $|U| \sqcup_{|W|} |V|$. ☐

Corollary 10.4.10. $C^\bullet_M$ *is a Nisnevich sheaf for every* R*-module* M*, by Prop. 10.4.7 and Lem. 10.4.9.*

Note. The argument for Lem. 10.4.9 given in [MV24, 4.1] seems insufficient, since one cannot identify the singular cohomology groups of the domain of an étale morphism of analytic spaces with the cohomology groups of its image in general. A similar inaccuracy seems to emerge at the proof provided in [MV24] of the following statement. In this case, the cohomology of the image and domain of a proper morphism are identified.

Lemma 10.4.11. *Assume that k is a trivially valued field, which allows resolutions of singularities. The functor* $|(\text{-})^{\mathrm{an}}|\colon \mathrm{Sm}^{\mathrm{sep}}_{/k} \to \mathrm{Top} \xrightarrow{y} \mathrm{Shv}_{\mathrm{open}}(\mathrm{Top})$ *sends cdh-coverings to effective epimorphisms.*

Proof. By Definition, every cdh-cover can be factored into Nisnevich coverings and proper cdh-coverings. By [MVW06, p.97] every proper cdh-covering of an $X \in \mathrm{Sm}^{\mathrm{sep}}_{/k}$ admits an refinement by blow-ups along smooth centers. Consider the associated square

$$\begin{tikzcd}
Z' \arrow[r] \arrow[d] & X' \arrow[d] \\
Z \arrow[r, hook] & X
\end{tikzcd}$$

where Z is the smooth center of the blow-up. Then, all schemes in this square are smooth and separated over k. Thus, their analytifications are contractible by Prop. 10.1.2. In particular, $|X^{\mathrm{an}}| \cong \mathrm{hocolim}(|Z^{\mathrm{an}}| \coprod |X'^{\mathrm{an}}| \to |\mathrm{X}^{\mathrm{an}}|)$. ☐

Corollary 10.4.12. $|(\text{-})^{\mathrm{an}}|$ *induces a geometric morphism in* $\mathcal{LT}\mathrm{op}$

$$|(\text{-})^{\mathrm{an}}|\colon\ \mathrm{Shv}_{\mathrm{cdh}}(\mathrm{Sch}^{\mathrm{lft}}_{/k}) \to \mathrm{Shv}_{\mathrm{open}}(\mathrm{Top}).$$

Proof. By the proof of Thm. 2.3.23, there is a geometric morphism

$$|(\text{-})^{\mathrm{an}}|\colon\ \mathrm{Shv}_{\mathrm{cdh}}(\mathrm{Sm}^{\mathrm{sep}}_{/k}) \to \mathrm{Shv}_{\mathrm{open}}(\mathrm{Top}).$$

However, the category $\mathrm{SmAff}^{\mathrm{fty}}$ of smooth affine schemes finite type over k is a full subcategory of $\mathrm{Sm}^{\mathrm{sep}}_{/k}$, since all affine schemes are separated. Similar to Prop. C.4.10, the all cdh-sheaves are determined by their values on affine schemes. Since k allows resolutions of singularities, Thm. 10.3.9 yields the following equivalences

$$\mathrm{Shv}_{\mathrm{cdh}}(\mathrm{Sch}^{\mathrm{lft}}_{/k}) \simeq \mathrm{Shv}_{\mathrm{cdh}}(\mathrm{Aff}^{\mathrm{fty}/k}) \simeq \mathrm{Shv}_{\mathrm{cdh}}(\mathrm{SmAff}^{\mathrm{fty}}_{/k}) \simeq \mathrm{Shv}_{\mathrm{cdh}}(\mathrm{Sm}^{\mathrm{sep}}_{/k}).$$ ☐

Corollary 10.4.13. $C^\bullet(|(\text{-})^{\mathrm{an}}|, M)\colon (\mathrm{Sch}^{\mathrm{lft}}_{/k})^{\mathrm{op}} \to \mathrm{D}(R)$ *is a cdh-sheaf for every R-module M.*

Thus, by considering the commutative triangle

$$\begin{tikzcd}
\mathrm{Shv}_{\mathrm{cdh}}(\mathrm{Sch}^{\mathrm{lft}}_{/k}; \mathrm{D}(R)) \arrow[rr, hook, "i"] \arrow[dr, "\Gamma"'] & & \mathcal{P}(\mathrm{Sch}^{\mathrm{lft}}_{/k}; \mathrm{D}(R)) \arrow[dl, "\Gamma"] \\
& \mathrm{D}(R) &
\end{tikzcd}$$

$\mathrm{map}_{\mathrm{cdh}}(L_{\mathrm{cdh}} y(X), C^\bullet_M) \simeq \mathrm{map}_{\mathcal{P}}(y(X), C^\bullet_M) \simeq C^\bullet_M(X)$ is an equivalence for all $X \in \mathrm{Sch}^{\mathrm{lft}}_{/k}$.

Equivalently, the canonical map $C^\bullet_M(X) \to R\Gamma_{\mathrm{cdh}}(X, C^\bullet_M)$ is a quasi-isomorphism for every $X \in \mathrm{Sch}^{\mathrm{lft}}_{/k}$. Classically, this fact is expressed by saying that $C^\bullet_M$ is *quasifibrant* (in the injective local model) (cf.[Cor+05, 3.3]). The same argument for any presheaf $P \in \mathcal{P}(\mathrm{Sch}^{\mathrm{lft}}_{/k})$ which satisfies cdh-excision recovers the well-known Theorem [Cor+05, 3.4].

The main observation in [MV24] is that the previous comparisons provide that the canonical morphism $A[0] \to C^\bullet(|X^{\mathrm{an}}|; A)$ induces an equivalence for every $A \in \mathrm{Ab}$.

Theorem 10.4.14 ([MV24, 4.3])**.** *Assume that k admits (strong) resolutions of singularities. For any abelian group A there is an equivalence in* $\mathrm{Shv}_{\mathrm{cdh}}(\mathrm{Sch}^{\mathrm{lft}}_{/k}; \mathrm{D}(\mathbb{Z}))$

$$R\Gamma_{\mathrm{cdh}}(-, \underline{A}) \xrightarrow{\simeq} R\Gamma_{\mathrm{cdh}}(-, C^\bullet_A) \simeq C^\bullet_{\mathrm{sing}}(|(-)^{\mathrm{an}}|; A)$$

where $\underline{A}$ is the constant sheaf. In particular, for every scheme X locally of finite type over k there is an isomorphism of cohomology groups

$$H^n_{\mathrm{sing}}(|X^{\mathrm{an}}|; A) \simeq H^{\mathrm{n}}_{\mathrm{cdh}}(X, \underline{A}) \qquad \forall\, n \geq 0.$$

Proof. Since k admits resolutions of singularities, it suffices to prove the statement after restricting to $(\mathrm{Sm}^{\mathrm{lft}}_{/k})^{\mathrm{op}}$, since cdh-sheaves on $\mathrm{Sch}^{\mathrm{lft}}_{/k}$ are uniquely determined by their values on smooth schemes (cf.Prop. 10.3.9). If X is a smooth scheme, the space $|X^{\mathrm{an}}|$ is contractible by Prop. 10.1.2 Hence, $C^\bullet(|X^{\mathrm{an}}|, A) \simeq A$. Moreover, constant Zariski sheaves are flasque in the sense of [Har77, p.207]. Further, all constant Nisnevich sheaves are $\mathbb{A}^1$-local and admit transfers. Consequently, the previous comparisons provide the following quasi-isomorphisms for every $X \in \mathrm{Sm}^{\mathrm{lft}}_{/k}$

$$C^\bullet_{\mathrm{sing}}(|X^{\mathrm{an}}|, A) \simeq \bigoplus_{x \in \pi_0 X} A[0] \overset{[\mathrm{Har77,\ III.2.5}]}{\simeq} R\Gamma_{\mathrm{Zar}}(X, \underline{A}) \overset{10.2.26}{\simeq} R\Gamma_{\mathrm{Nis}}(X, \underline{A}) \overset{10.3.10}{\simeq} R\Gamma_{\mathrm{cdh}}(X, \underline{A}).$$

□

Corollary 10.4.15. *Assume k allows resolutions of singularities. By Thm. 4.6.9 the composition* |-| ∘ (-)$^{\mathrm{an}}$ *induces an equivalence of profinite shape functors*

$${}^{\mathrm{ab}}\widehat{\Pi}_\infty^{\mathrm{Shv}_{\mathrm{open}}(\mathrm{Top})} \circ |(\text{-})^{\mathrm{an}}| \overset{\simeq}{\Longrightarrow} {}^{\mathrm{ab}}\widehat{\Pi}_\infty^{\mathrm{Shv}_{\mathrm{cdh}}(\mathrm{Sch}^{\mathrm{lft}}_{/k})}.$$

Corollary 10.4.16. *Assume k allows resolutions of singularities. If X is a smooth scheme locally of finite type over k or a quasi-projective algebraic k-variety, then the abelian profinite shape its cdh topos is given by*

$$\widehat{\Pi}^{\mathrm{ab}}_\infty \mathrm{Shv}_{\mathrm{cdh}}(X) \simeq \widehat{\mathrm{Sing}(|X^{\mathrm{an}}|)}^{\mathrm{ab}}$$

where $\widehat{(\text{-})}^{\mathrm{ab}} : \mathrm{Ani} \to \mathrm{Pro}(\mathrm{Ani}^{\pi,\mathrm{ab}}$ *denotes the abelian profinite completion.*
In case, X is a smooth separated scheme over k, the abelian profinite shape of X vanishes

$$\widehat{\Pi}^{\mathrm{ab}}_\infty \mathrm{Shv}_{\mathrm{cdh}}(X) \simeq *.$$

Conjecture 10.4.17. Thm. 10.4.14 extends to an equivalence of principal G-bundles for all finite (discrete) groups G.

This would allow to lift the restriction to abelian profinite shapes in the Corollaries above and exhibit that the profinite cdh-shape of a smooth separated scheme X over k is represented the homotopy type of $|X^{\mathrm{an}}|$.

I look forward to applications of the constancy of the profinite cdh-shape in motivic homotopy theory and K-theory comparable to the étale complex realization functor (cf.[CE18], [BH20]).

A Appendix A - Recollections on ∞-Categories

A.1 Functoriality of the Slice construction

The following section establishes key properties of slice categories, formulated so as to naturally extend the theory developed in [Lan21].

Lemma A.1.1. *Let $\mathcal{C}$ be ∞-category. Then, evaluating at the target yields a cocartesian fibration $ev_1 : \mathcal{C}^{\Delta^1} \to \mathcal{C}$ and straightening ev_1 produces the slicing functor*

$$\mathcal{C}_{/-} : \mathcal{C} \to \widehat{\mathbf{Cat}}_\infty, \quad x \mapsto \mathcal{C}_{/x}$$

Here, a morphism $f\colon x \to y$ in $\mathcal{C}$ is sent to the postcomposition functor

$$f_!\colon \mathcal{C}_{/x} \to \mathcal{C}_{/y},\ (g\colon z \to y) \mapsto (f \circ g\colon z \to y).$$

Proof. The statement readily follows from [HTT, 2.4.7.12], but the argument given here is somewhat more direct. Expanding the definitions in [Lan21], the task reduces to verifying that every lifting problem

$$\begin{tikzcd}
\{0\} \arrow[r, "g"] \arrow[d, hook, "\cap"] & \mathcal{C}^{\Delta^1} \arrow[d, "\mathrm{ev}_1"] \\
\Delta^1 \arrow[r, "f"] \arrow[ur, dashed, "s"] & \mathcal{C}
\end{tikzcd}$$

admits a solution s, which is an ev_1-cocartesian morphism. As explained in [Lan21, 1.3.30], the adjunction $- \times \Delta^1 \dashv \underline{\mathrm{Hom}}_{\mathbf{Set}^{\Delta^{\mathrm{op}}}}(\Delta^1, -)$ implies, that this amounts precisely to solving every lifting problem of the form

$$\begin{tikzcd}
\{1\} \times \Delta^1 \sqcup_{\{1\}\times\{0\}} \Delta^1 \times \{0\} \arrow[r, "f \sqcup_x g"] \arrow[d] & \mathcal{C} \\
\Delta^1 \times \Delta^1 \arrow[ur, dashed, "s"] &
\end{tikzcd}$$

so that the lift s corresponds to an ev_1-cocartesian morphism in C^{Δ^1}.
Then, the desired lift $s\colon \Delta^1 \times \Delta^1 \to \mathcal{C}$ for given edges $f\colon x \to y$ and $g\colon z \to x$ is induced by the square

$$\begin{tikzcd}
z \arrow[r, "f \circ g"] & y \\
z \arrow[u, "id_z"] \arrow[r, "g"] & x \arrow[u, "f"']
\end{tikzcd}$$

We claim that the corresponding morphism $s\colon \Delta^1 \to \mathcal{C}^{\Delta^1}$ is ev_1-cocartesian.
Suppose s appears in a diagram

$$\begin{tikzcd}
\Delta^{0,1} \arrow[r] \arrow[rr, bend left, "s"] & \Lambda^n_0 \arrow[r, "g"] \arrow[d, hook, "i"] & \mathcal{C}^{\Delta^1} \arrow[d, "\mathrm{ev}_1"] \\
 & \Delta^n \arrow[r] & \mathcal{C}
\end{tikzcd}$$

Then, by Definition, composing g with $ev_0\colon \mathcal{C}^{\Delta^n} \to \mathcal{C}$ yields a horn $\Delta^{0,1} \to \Lambda^n_0 \to \mathcal{C}$, where the image of $\Delta^{0,1}$ is $id_z\colon z \to z$. This means i factors as $\Lambda^n_0 \xrightarrow{d_1} \Delta^{n-1} \xrightarrow{s_0} \Delta^n$.

Let $j\colon \Delta^{n-1} \hookrightarrow \Lambda^n_0$ be the unique inclusion, such that $\Delta^{n-1} \overset{j}{\hookrightarrow} \Lambda^n_0 \xrightarrow{d_1} \Delta^{n-1}$ is the identity. Then, since $\mathcal{C}^{\Delta^1}$ is a simplicial set, there is a lift such that the following diagram commutes

$$\begin{tikzcd}
\Lambda^n_0 \arrow[r, "g"] \arrow[d, "d_1"'] & \mathcal{C}^{\Delta^1} \arrow[dd] \\
\Delta^{n-1} \arrow[ur, "g\circ j"] \arrow[d, "s_0"'] & \\
\Delta^n \arrow[r] \arrow[uur, dashed] & \mathcal{C}
\end{tikzcd}$$

By [Lan21, Lem.2.5.30], $\mathcal{C}_{/x}$ is equivalent to the fiber of x under ev_1. Therefore, straightening $ev_1 : \mathcal{C}^{\Delta^1} \to \mathcal{C}$ produces the functor

$$\mathcal{C}_{/-}\colon \mathcal{C} \to \widehat{\mathrm{Cat}}_\infty,\ x \mapsto \mathcal{C}_{/x}$$

As explained in [Lan21, see 3.3.1] and [Lan21, 3.2.16], the map $f_!\colon \mathcal{C}_{/x} \to \mathcal{C}_{/y}$ induced by a morphism $f\colon x \to y$ is defined as the composition

$$\mathcal{C}_{/x} \xrightarrow{s^{-1}} \mathrm{Fun}^{\mathrm{cc}}_f(\Delta^1, \mathcal{C}^{\Delta^1}) \xrightarrow{t} \mathcal{C}_{/y}$$

where s^{-1} is the essentially unique inverse of the trivial fibration $s\colon \mathrm{Fun}^{\mathrm{cc}}_f(\Delta^1, \mathcal{C}^{\Delta^1}) \to \mathcal{C}_{/x}$. By its construction in [Lan21, 3.3.1], $f_!$ sends an $z \xrightarrow{g} x$ of $\mathcal{C}_{/x}$ to some choice of ev_1-cocartesian lift

$$\begin{tikzcd}
f_!(z) \arrow[r, "f_!(g)"] & y \\
x' \arrow[u, "f_!"] \arrow[r, "g"] & x \arrow[u, "f"']
\end{tikzcd}$$

However, $f_!(z)$ and $f_!(g)$ are unique up to contractible choice, because $f_!$ is. Hence, $f_!(z) \simeq z$ and $f_!(g) \simeq f \circ g$ by the above. ☐

The next Lemma provides a direct description of mapping spaces in slice categories in terms of mapping spaces of the ambient category.

Lemma A.1.2 ([HTT, Lem.5.5.5.12]). *Let $\mathcal{C}$ be a ∞-category and $f\colon x \to y$, $g\colon z \to y$ be two morphisms in $\mathcal{C}$. The space of maps from f to g is computed as the homotopy fiber*

$$\begin{tikzcd}
\mathrm{map}_{\mathcal{C}_{/y}}(x,z) \arrow[r] \arrow[d] & \mathrm{map}_{\mathcal{C}}(x,z) \arrow[d, "g_*"] \\
\Delta^0 \arrow[r, "f"] & \mathrm{map}_{\mathcal{C}}(x,y)
\end{tikzcd}$$

Proof. A slightly different variant of the proof of [HTT, Lem. 5.5.5.12] is presented by relying on results from [Lan21].

Recall from [Lan21, 2.5.27] that $\mathcal{C}_{/y} \simeq \mathcal{C}^{/y}$ for every $y \in \mathcal{C}$. By [Lan21, Cor.2.5.31] and the pasting law for pullbacks, all squares in the following diagram are pullbacks

$$\begin{tikzcd}
\operatorname{map}_{\mathcal{C}}(x,z) \arrow[r] \arrow[d, "g_*"] & \mathcal{C}_{/z} \arrow[d, "g_!"] \\
\operatorname{map}_{\mathcal{C}}(x,y) \arrow[r] \arrow[d] & \mathcal{C}_{/y} \arrow[d] \\
\Delta^0 \arrow[r, "x"] & \mathcal{C}
\end{tikzcd}$$

Recall from [Lan21, Thm.1.4.23], that the composition $\{0\} \to \Delta^1 \xrightarrow{g} \mathcal{C}$ induces a trivial fibration $\mathcal{C}^{/g} \to \mathcal{C}^{/z}$, since $\{0\} \to \Delta^1$ is left-anodyne. Therefore, $\mathcal{C}^{/g} \to \mathcal{C}^{/z}$ is coinitial[83] by [Lan21, Lem.4.4.11]. Hence, $\operatorname{map}_{\mathcal{C}}(x,z) \simeq \{x\} \times_{\mathcal{C}} \mathcal{C}^{/g}$.
Taking the pullback along $f\colon \Delta^0 \to \operatorname{map}_{\mathcal{C}}(x,y)$ leaves us a diagram of pullback squares

$$\begin{tikzcd}
\{f\} \times_{\mathcal{C}} \mathcal{C}^{/g} \arrow[r] \arrow[d] & \operatorname{map}_{\mathcal{C}}(x,z) \arrow[r] \arrow[d] & \mathcal{C}^{/g} \arrow[d] \\
\{f\} \arrow[r] & \operatorname{map}_{\mathcal{C}}(x,y) \arrow[r] & \mathcal{C}^{/y}
\end{tikzcd}$$

Now, $\mathcal{C}^{/g} \simeq (\mathcal{C}^{/y})^{/z}$ can be identified, since

$$\begin{aligned}
&\operatorname{Fun}(K, (\mathcal{C}^{/y})^{/z}) \cong \operatorname{Fun}(K \diamond \{z\}, \mathcal{C}^{/y}) \cong \operatorname{Fun}(K \diamond \{z\} \diamond \{y\}, \mathcal{C}) \\
&\overset{\text{[Lan21, 2.5.19]}}{\simeq} \operatorname{Fun}(K \diamond \{g\}, \mathcal{C}) \cong \operatorname{Fun}(K, \mathcal{C})
\end{aligned}$$

for every $K \in \mathbf{Set}^{\Delta^{\mathrm{op}}}$. This implies

$$\operatorname{map}_{\mathcal{C}^{/y}}(x,z) \overset{\text{[Lan21, 2.5.31]}}{\cong} \{f\} \times_{\mathcal{C}^{/y}} (\mathcal{C}^{/y})^{/z} \cong \{f\} \times_{\mathcal{C}^{/y}} \mathcal{C}^{/g}. \qquad \square$$

Corollary A.1.3. *Let $\mathcal{C}$ be a ∞-category with finite limits. Let $1_{\mathcal{C}} \in \mathcal{C}$ be a terminal object. For any object $x, u \in \mathcal{C}$ there is an equivalence*

$$\operatorname{map}_{\mathcal{C}_{/u}}(1_u, x \times u) \simeq \operatorname{map}_{\mathcal{C}}(u, x)$$

where $1_u := \mathrm{id}_u$ is the terminal object of $\mathcal{C}_{/u}$ and $x \times u$ is the canonical map $\mathrm{pr}_u\colon x \times u \to u$. Thus, the

Proof. Since map preserves limits in the second variable, there are equivalences

$$\begin{aligned}
&\operatorname{map}_{\mathcal{C}_{/u}}(1_u, x \times u) \overset{A.1.2}{\simeq} \operatorname{map}_{\mathcal{C}}(u, x \times u) \times_{\operatorname{map}_{\mathcal{C}}(u,u)} \{\mathrm{id}_u\} \\
&\simeq \operatorname{map}_{\mathcal{C}}(u, x) \times_{\operatorname{map}_{\mathcal{C}}(u,1_{\mathcal{C}})} \operatorname{map}_{\mathcal{C}}(u,u) \times_{\operatorname{map}_{\mathcal{C}}(u,u)} \{\mathrm{id}_u\} \simeq \operatorname{map}_{\mathcal{C}}(u, x). \qquad \square
\end{aligned}$$

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[83] i.e. limits in $\mathcal{C}^{/z}$ are equally given by limits in $\mathcal{C}^{/g}$.

Next, we show that taking pullbacks gives rise to functors on slice categories if the ambient category admits enough pullbacks.

Lemma A.1.4. *Let $\mathcal{C}$ be an ∞-category and $\mathrm{ev}_1\colon \mathcal{C}^{\Delta^1} \to \mathcal{C}$ the evaluation on the target. Let $F : \Delta^1 \to \mathcal{C}^{\Delta^1}$ be the morphism corresponding to the following square in $\mathcal{C}$*

$$\begin{tikzcd}
x' \arrow[r] \arrow[d] & y' \arrow[d] \\
x \arrow[r] & y
\end{tikzcd}$$

Then, F is a ev_1-cartesian morphism if and only if the square above is a pullback in $\mathcal{C}$.
In particular, ev_1 is a bicartesian fibration if and only if $\mathcal{C}$ admits all pullbacks.

Sketch of a Proof. Recall from [Lan21, 3.1.2], that the morphism F corresponding to the square above is ev_1-cartesian iff $\varphi\colon \mathcal{C}^{\Delta^1}_{/F} \to \mathcal{C}^{\Delta^1}_{/g} \times_{\mathcal{C}_{/y'}} \mathcal{C}_{/f}$ is a trivial fibration. The strategy is to define an ∞-category $\mathcal{D}$, which functions as the slice of $\mathcal{C}^{\Delta^1}$ over the lower right corner of the square F. Then one shows, that φ factors through $\mathcal{D}$ and both $\mathcal{C}^{\Delta^1}_{/F} \to \mathcal{D}$ and $\mathcal{D} \to \mathcal{C}^{\Delta^1}_{/g} \times_{\mathcal{C}_{/y'}} \mathcal{C}_{/f}$ are trivial fibrations if and only if F yields a pullback diagram in $\mathcal{C}$. See [HTT, 6.1.1.1] for details. ☐

Corollary A.1.5. *Let $\mathcal{C}$ be an ∞-category with pullbacks, and let $f\colon x \to y$ be a morphism in $\mathcal{C}$. Then, the functor $f_!$ of Lemma A.1.1 admits the following right adjoint, given on objects by pullback along f:*

$$f^*\colon \mathcal{C}_{/y} \to \mathcal{C}_{/x},\ (z \to y) \mapsto (z \times_y x \to x)$$

Proof. Let $f_!\colon \Delta^1 \xrightarrow{f} \mathcal{C} \to \mathrm{Cat}_\infty$ be the functor associated to $f\colon x \to y$. Taking the pullback

$$\begin{tikzcd}
\mathcal{E} \arrow[r] \arrow[d, "p"] & \mathcal{C}^{\Delta^1} \arrow[d, "\mathrm{ev}_1"] \\
\Delta^1 \arrow[r] & \mathcal{C}
\end{tikzcd}$$

provides us with a bicartesian fibration $p\colon \mathcal{E} \to \Delta^1$. Thus, $f_!$ admits a right adjoint, which is deduced from $\mathrm{Str}(p)\colon (\Delta^1)^{\mathrm{op}} \to \mathrm{Cat}_\infty$ i.e. the straightening of p viewed as cartesian fibration (see [Lan21, 5.1.2]). More precisely, $\mathrm{Str}(p)$ factors through $\mathrm{Str}(ev_1)\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ and therefore the right adjoint of $f_!$ is given by the composition

$$f^*\colon \mathcal{C}_{/y} \simeq \mathcal{C}^{\Delta^1}_y \to \mathcal{C}^{\Delta^1}_x \simeq \mathcal{C}_{/x}$$

which is exactly the functor obtained by restricting $\mathcal{C}_{/y} \times (\Delta^1)^{\mathrm{op}} \to \mathcal{C}^{\Delta^1}$ to $\mathcal{C}/y \times \{0\}$ (see [Lan21, 3.3.1]). The restriction of $\mathcal{C}/y \times (\Delta^1)^{\mathrm{op}} \to \mathcal{C}^{\Delta^1}$ to any object $(y' \to y)$ of $\mathcal{C}_{/y}$, on the other hand, corresponds to a ev_1-cartesian lift of $f^{\mathrm{op}}\colon y \to x$. Thus, the image of $(y' \to y)$ under f^* is the left vertical arrow in a square

$$\begin{tikzcd}
x' \arrow[r] \arrow[d] & y' \arrow[d] \\
x \arrow[r, "f"] & y
\end{tikzcd}$$

corresponding to a ev_1-cartesian lift of f. By Lem. A.1.4 one concludes $f^*(y') \simeq y' \times_y x$. ☐

Slice categories admit just as many colimits as the ambient category.

The forgetful functor $f_!$ does not preserve limits in general. For example, if $\mathcal{C}$ has finite limits and $x \in \mathcal{C}$ is not a terminal object, then $f_!(id_x) \simeq x$ is not terminal, although $id_x \in \mathcal{C}_{/x}$ is a terminal object. However, for diagrams over weakly contractible simplicial sets, the forgetful functor not only preserves limits, but acts conservatively with respect to them.

Proposition A.1.6 ([HTT, 4.4.2.9])**.** *Let $f : \mathcal{C} \to \mathcal{D}$ be a right fibration of ∞-categories, and let $p : K \to \mathcal{C}$ be a diagram with K weakly contractible. Then,*

1) *Let $\mathcal{C}$ $\bar{p} : \Delta^0 \star K \to \mathcal{C}$ be an extension of p. Then $\bar{p}$ is a limit if and only if $f \circ \bar{p}$ is a limit for $f \circ p$.*
2) *If there is a limit $\bar{q} : \Delta^0 \star K \to \mathcal{D}$ of $f \circ p$, then there exists an extension $\bar{p}$: $\Delta^0 \star K \to \mathcal{C}$ of p such that $\bar{q} = f \circ \bar{p}$ and therefore $\bar{p}$ is a limit by* 1.

In particular, f preserves limits over weakly contractible diagrams and limits in $\mathcal{C}$ can be detected by applying f.

The proof-strategy for Prop. A.1.6 is lent from [HTT, 4.4.2.9]. Our arguments however will only depend on material in [Lan21]. A crucial Lemma to proving Prop. A.1.6 is the following generalization of [Lan21, 1.4.23], which will not be proved here.

Lemma A.1.7 ([HTT, 4.4.2.8])**.** *Let $K \subset \Delta^0 \star K \xrightarrow{\bar{p}} \mathcal{C} \xrightarrow{f} \mathcal{D}$ be composable maps, where $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories and K is weakly contractible. Then, the induced map*

$$\mathcal{C}_{/\bar{p}} \to \mathcal{C}_{/p} \times_{\mathcal{D}_{fp}} \mathcal{C}_{f\bar{p}}$$

is a trivial fibration, where $p := \bar{p}|_K$.

Proof of Prop. A.1.6. By [Lan21, 4.3.11][84] an extension $\bar{p}$: $\Delta^0 \star K \to \mathcal{C}$ of a functor p is a limit if and only if $\mathcal{C}_{/\bar{p}} \to \mathcal{C}_{/p}$ is a trivial fibration.
1. Consider the following diagram

$$\begin{tikzcd}
\mathcal{C}_{/\bar{p}} \arrow[r, "\phi"] & \mathcal{C}_{/p} \times_{\mathcal{D}_{/fp}} \mathcal{D}_{/f\bar{p}} \arrow[r, "\psi'"] \arrow[d, "\Theta'"] & \mathcal{C}_{/p} \arrow[d, "\theta"] \\
 & \mathcal{D}_{/f\bar{p}} \arrow[r, "\psi"] & \mathcal{D}_{/fp}
\end{tikzcd}$$

where ϕ is the trivial fibration of Lem. A.1.7.
To establish the "if"-direction, suppose $f\bar{p}$ is a limit i.e. ψ' is a trivial fibration. In this case, ψ is a trivial fibration, since trivial fibrations are stable under pullback and therefore $\psi \circ \phi$ is a trivial fibration.

For the converse, suppose $\bar{p}$ is a limit i.e. $\psi' \circ \phi$ is a trivial fibration. Since trivial fibrations satisfy the 2-out-of-3 property, ψ' is a trivial fibration. To prove that Θ is a trivial fibration, observe that the following two lifting problems are equivalent

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[84] In [HTT], limits are defined via this property.

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & \mathcal{C}_{/p} \arrow[d] & & \partial\Delta^n \star K \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & \mathcal{D}_{/fp} & \Leftrightarrow & \Delta^n \star K \arrow[r] \arrow[ur, dashed] & \mathcal{D} \end{tikzcd}$$

By Quillen's Theorem A [Lan21, 4.4.20], the monomorphism $i\colon \partial\Delta^n \star K \longrightarrow \Delta^n \star K$ is cofinal, since for every $d \in \Delta^n \star K$ the slice $(\partial\Delta^n \star K)_{d/}$ is equivalent to a slice over K, and hence weakly contractible. It follows from [Lan21, 4.4.4] that i is right anodyne. Thus, the above lifting problem admits a solution because $\mathcal{C} \to \mathcal{D}$ is a right fibration. Consequently, Θ, and therefore Θ', are trivial fibrations. Finally, by commutativity of the diagram and the 2-out-of-3 property for trivial fibrations ψ' is also a trivial fibration.

2. To find an extension $\bar{p}\colon \Delta^0 \star K \to \mathcal{C}$ of p with $\bar{q} \simeq f \circ \bar{p}$ amounts to solving the following lifting problem.

$$\begin{tikzcd} K \arrow[r] \arrow[d, hook] & \mathcal{C} \arrow[d, "f"] \\ \Delta^0 \star K \arrow[r] \arrow[ur, dashed] & \mathcal{D} \end{tikzcd}$$

By the reasoning above, the inclusion $K \subset \Delta^0 \star K$ is right-anodyne, since it its induced by $\emptyset = \partial\Delta^0 \subset \Delta^0$. Thus, this lifting problem admits a solution. ☐

As a direct follow up we get the following Corollary.

Corollary A.1.8. *Let $\mathcal{C}$ be an ∞-category with finite limits, $x \in \mathcal{C}$ and $t\colon x \to 1_{\mathcal{C}}$ a morphism the terminal object. A diagram $p\colon \Lambda^2_2 \to \mathcal{C}_{/x}$ is a pullback if and only if $t_! \circ p$ is a pullback.*

The following Observation will be helpful to identify certain squares in § 3.3 as pullbacks in slice categories.

Observation A.1.9. Let $\mathcal{C}$ be a ∞-category with finite limits. Let $x \in \mathcal{C}$ be some object and $f\colon x \to 1_{\mathcal{C}}$ a morphism to a terminal object. We have an adjunction $\mathcal{C} \underset{f^*}{\overset{f_!}{\leftrightarrows}} \mathcal{C}_{/x}$. Now, for every $y \xrightarrow{g} x$ in $\mathcal{C}_{/x}$ there is a square in $\mathcal{C}_{/x}$

$$\begin{tikzcd} g \arrow[r, "\epsilon_y"] \arrow[d] & f^*y \arrow[d] \\ 1_x \arrow[r, "\epsilon_x"] & f^*x \end{tikzcd}$$

where the horizontal arrows are induced by the unit of the adjunction, since $f^*(y) \simeq f^*f_!(g)$. By applying Cor. A.1.8, we recognize that this square has to a pullback square, as $f_!\colon \mathcal{C}_{/x} \to \mathcal{C}$ sends the it to the following pullback square in $\mathcal{C}$.

$$\begin{tikzcd} y \arrow[r] \arrow[d] & y \times x \arrow[d] \\ x \arrow[r] & x \times x \end{tikzcd}$$

From this, one deduces that the slicing construction enjoys "1-naturality", giving rise to commutative diagrams for every morphism.

Lemma A.1.10. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a finite limit preserving functor between ∞-categories. Then for every morphism $f\colon x \to y$ in $\mathcal{C}$, there is a commutative diagram in $\widehat{\mathrm{Cat}}_\infty$*

$$\begin{tikzcd}
\mathcal{C}_{/y} \arrow[r, "f^*"] \arrow[d, "F"] & \mathcal{C}_{/x} \arrow[d, "F"] \\
\mathcal{D}_{/F(y)} \arrow[r, "F(f^*)"] & \mathcal{D}_{/F(x)}
\end{tikzcd}$$

The naturality of this construction persists in higher levels, meaning that there is an natural transformation

$$\alpha\colon \mathcal{C}^*_{/-} \to \mathcal{D}^*_{/-} \circ F^{\mathrm{op}}.$$

of functors from $\mathcal{C}^{\mathrm{op}}$ to Cat_∞.

Remark A.1.11. The naturality of the slicing operation of Lem. A.1.1, which sends a ∞-category $\mathcal{C}$ to $\mathcal{C}_{/-} \in \mathrm{Fun}(\mathcal{C}, \widehat{\mathrm{Cat}}_\infty)$ persists in higher levels. However, to define such a functor with domain $\widehat{\mathrm{Cat}}_\infty$, one would have to establish a notion $(\infty,2)$-categories. Such technicalities shall not be a concern of this thesis. Throughout the remaining thesis it will simply be assumed that slicing yields such an appropriate functor.

A.2 Locally cartesian closed ∞-Categories

As will be shown in this section, functors introduced above give rise to mapping spaces **internal** to slice ∞-categories, satisfying the usual tensor–hom adjunction. In suitably well-behaved contexts – such as categories of presheaves of small categories – these internal mapping spaces also exist globally on the ambient category from which the slice categories arise.

To describe this, a notion of internal product on slice categories is needed.

Definition A.2.1. Let $\mathcal{C}$ be an ∞-category with pullbacks and let $f\colon x \to y$ be a morphism. Define $x \times_y - \ := \ f_! \circ f^*$, and refer to this as the **product functor** in $\mathcal{C}_{/x}$.

Recall, that an ∞-category $\mathcal{C}$ with finite products is cartesian closed, if it admits internal mapping objects $\underline{\mathrm{Hom}}(c,d)$ for all $c, d \in \mathcal{C}$, and for each $c \in \mathcal{C}$ there is an adjunction

$$c \times - \ \dashv \ \underline{\mathrm{Hom}}(c, -).$$

A convenient way to introduce internal mapping spaces is to define them via this tensor–hom adjunction.

Definition A.2.2. An ∞-category $\mathcal{C}$ is **locally cartesian closed** if for every $y \in \mathcal{C}$ and every morphism $f\colon x \to y$, the product functor $x \times_y -\colon \mathcal{C}_{/x} \to \mathcal{C}_{/x}$ admits a right adjoint $\underline{\mathrm{Hom}}_{\mathcal{C}_{/x}}(f, -)$.

As expected, the existence of right adjoints to pullback functors ensures this property.

Lemma A.2.3. *Let $\mathcal{C}$ be a ∞-category with pullbacks, such that each pullback functor $f^*\colon \mathcal{C}_{/y} \to \mathcal{C}_{/x}$ for an edge $f\colon x \to y$, admits a right adjoint f_*.*
Then, $\mathcal{C}$ is locally cartesian closed and for every slice category $\mathcal{C}_{/y}$ its internal mapping object $\underline{\mathrm{Hom}}_{\mathcal{C}_{/y}}$ is given by

$$\underline{\mathrm{Hom}}(x \xrightarrow{f} y, -) \simeq f_* \circ f^*(-)$$

Proof. Let $f\colon x \to y$ be a morphism in $\mathcal{C}$. By assumption, there is an adjoint triple $f_! \dashv f^* \dashv f_*$ where $f_! : \mathcal{C}_{/x} \to \mathcal{C}_{/y}$ is the postcomposition with f. Then, the product of $x \xrightarrow{f} y$ and $z \xrightarrow{g} y$ in $\mathcal{C}_{/y}$ is given by $x \times_y z = f_! \circ f^*(g)$. Therefore, $f_* \circ f^*(-)$ is right adjoint to $x \times_y - = f_! \circ f^*$ and setting

$$\underline{\mathrm{Hom}}(x \xrightarrow{f} y, -) := f_* \circ f^*(-)$$

provides an internal mapping object for $\mathcal{C}_{/y}$. □

Since the condition of Lem. A.2.3 is frequently satisfied in practice, the following definition has become standard.

Definition A.2.4. Let $\mathcal{C}$ be a presentable category. Colimits in $\mathcal{C}$ are said to be **universal** if, for every morphism $f\colon x \to y$, the pullback functor $f^*\colon \mathcal{C}_{/y} \to \mathcal{C}_{/x}$ preserves small colimits or, equivalently[85], f^* admits a right adjoint $f_*\colon \mathcal{C}_{/x} \to \mathcal{C}_{/y}$.

Observation A.2.5. By Cor. A.1.5 every presentable category, whose colimits are universal, is locally and globally cartesian closed, since every presentable category has a terminal object.

In particular, all ∞-categories encountered in section § 2 have universal colimits.

A.3 Presentable ∞-categories are (co)tensored over Anima

Recall, that $\mathrm{Ani} \simeq \mathcal{P}(*)$ is the free cocompletion of the point $* = \Delta^0$. From the description of presheaves via Kan extensions (c.f.[Wag23, 6.30]) it follows that every $K \in \mathrm{Ani}$ can be written as colimit

$$K \cong \operatorname*{colim}_{k \in K} *$$

over its points $k\colon * \to K$. In consequence, every mapping space can be described as limit

$$\mathrm{map}_{\mathrm{Ani}}(K, L) \cong \mathrm{map}_{\mathrm{Ani}}(\operatorname*{colim}_{k \in K} *, L) \cong \operatorname*{map}_{k \in K}{}_{\mathrm{Ani}}(*, L) \cong \lim_{k \in K} L$$

As will be outlined in this section, these facts imply that bicomplete ∞-categories are tensored and cotensored over Ani.

Definition A.3.1. An ∞-category $\mathcal{C}$ is tensored over Ani if, for every $c \in \mathcal{C}$, the functor $\mathrm{map}_{\mathcal{C}}(c, -)\colon \mathcal{C} \to \mathrm{Ani}$ admits a left adjoint

$$- \cdot c\colon \mathrm{Ani} \to \mathcal{C}$$

which is functorial in c. In this case, $\mathcal{C}$ is further cotensored over Ani, if, for every $K \in \mathrm{Ani}$, the functor $K \cdot -\colon \mathcal{C} \to \mathcal{C}$ admits a right adjoint

$$(-)^K\colon \mathcal{C} \to \mathcal{C}.$$

Lemma A.3.2. *Any cocomplete ∞-category $\mathcal{C}$ is tensored over* Ani *by setting*

$$K \cdot c := \operatorname{colim} K \to \Delta^0 \xrightarrow{c} \mathcal{C}$$

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[85] See for example [Lan21, 5.2.4]

for each $c \in \mathcal{C}$. This assignment is functorial.

Proof. Since $\mathrm{map}_{\mathcal{C}}$ is continuous, there are equivalences for any $c, d \in \mathcal{C}$ and $K \in \mathrm{Ani}$

$$\mathrm{map}_{\mathcal{C}}(K \cdot c, d) \simeq \lim_K \mathrm{map}_{\mathcal{C}}(c, d) \simeq \mathrm{map}_{\mathrm{Ani}}(K, \mathrm{map}_{\mathcal{C}}(c, d)). \qquad \square$$

Lemma A.3.3. *Any complete and cocomplete ∞-category $\mathcal{C}$ is tensored and cotensored over* Ani *by setting*

$$c^K := \lim K \to \Delta^0 \xrightarrow{c} \mathcal{C}$$

Proof. For given $K \in \mathrm{Ani}$ and $c \in \mathcal{C}$, let $\delta_c^K : K \to \Delta^0 \to \mathcal{C}$ be the constant functor used above. The canonical adjunction between the (co)limit and the constant functor δ provide the following equivalences, natural in $c, d \in \mathcal{C}$

$$\mathrm{map}_{\mathcal{C}}(K \cdot c, d) \simeq \mathrm{map}_{\mathrm{Fun}(K,\mathcal{C})}(\delta_c, \delta_d) \simeq \mathrm{map}_{\mathcal{C}}(c, d^K). \qquad \square$$

In suitable situations – specifically those treated in the next section – the cotensors c^K coincide with internal mapping spaces. This will be clarified in Lem. 2.3.28.

A.4 Supplements to Section 2

Proof of Obs. 2.9.4:
Pick a finite subcovering for every covering in τ and let σ be the collection of those finite subcoverings. One readily checks that σ is a pretopology on $\mathcal{C}$ because

1) σ contains all equivalences;

2) if $\{x_{i_0} \to x\}_{i_0 \in I_0} \subset \{x_i \to x\}_{i \in I}$ is a finite subcover and $y \to x$ a morphism in $\mathcal{C}$, then $\{x_{i_0} \times_x y \to y\}_{I_0} \subset \{x_i \times_x y \to y\}_I$ is a finite subcover.

3) compositions of finite subcovers are finite subcovers.

Now, every sieve $\mathcal{C}^{\tau}_{/x} \subset \mathcal{C}_{/x}$, which contains a covering $\{x_i \to x\} \in \mathrm{Cov}_\tau(x)$ also contains a finite subcover $\{x_{i_0} \to x\}$ thereof, thus is a covering sieve wrt. σ. Likewise, every sieve $\mathcal{C}^{\sigma}_{/x} \subset \mathcal{C}_{/x}$, which contains a covering $\{x_{i_0} \to x\} \in \mathrm{Cov}_\sigma(x)$, is a covering sieve wrt. τ, since by Definition $\{x_{i_0} \to x\} \in \mathrm{Cov}_\tau(x)$.

A.5 Supplements on Recollements

Axiomatized initially in [AGVa, IV.9], the theory of *recollements* formalizes 4-functor formalism, naturally extending to a full 6-functor formalism in the stable case. Following [HA; Sha22], the notion of a recollement extends to the ∞-categorical setting, where a complete classification for ∞-topoi is provided by [HA, A.8.16].

Definition A.5.1 ([Sha22, 2.1])**.** Let $\mathcal{X}$ be an ∞-category with finite limits and let $\mathcal{U}, \mathcal{Z} \subset \mathcal{X}$ be full subcategories that are stable under equivalences. $(\mathcal{U}, \mathcal{Z})$ is a **recollement** of $\mathcal{X}$ if the inclusion functors $j_* : \mathcal{U} \subset \mathcal{X}$ and $i_* : \mathcal{Z} \subset \mathcal{X}$ admit left exact left adjoints[86] j^* and i^* such that

1) $j^* i_*$ is equivalent to the constant functor at the terminal object $1_{\mathcal{U}}$ of $\mathcal{U}$.
2) j^* and i^* are jointly conservative, i.e., if $f : x \to y$ is a morphism in $\mathcal{X}$ such that $j^* f$ and $i^* f$ are equivalences, then f is an equivalence.

$\mathcal{U}$ is called the **open** part of the recollement, $\mathcal{Z}$ the **closed** part of the recollement, and $i^* j_*$ the **gluing functor**.

Every recollement yields a functorial fracture-square already in this general setting.

Proposition A.5.2 ([Sha22, 2.2]). *For any recollement $(\mathcal{U}, \mathcal{Z})$ of an ∞-category $\mathcal{X}$ with finite limits, there are pullback squares of endofunctors*

$$\begin{tikzcd}
\mathrm{id}_{\mathcal{X}} \arrow[r, "\eta_i"] \arrow[d, "\eta_j"'] & i_* i^* \arrow[d, "i_* i^* \bullet \eta_j"] \\
j_* j^* \arrow[r, "\eta_i \bullet j_* j^*"] & i_* i^* j_* j^*
\end{tikzcd} \qquad (\boxed{\mathfrak{R}})$$

where η_i and η_j are the units of the adjunctions $i^* \dashv i_*$ and $j^* \dashv j_*$.

Proof. Since j^* and i^* are left-exact and jointly conservative, it suffices to show that $\boxed{\mathfrak{R}}$ is a pullback square after applying i^* and j^*, respectively.

1. Case: i^*
Since $i^* i_* \simeq \mathrm{id}_{\mathcal{Z}}$ (i^* is a reflection), the pullback can pointwise be calculated as

$$i^* j_* j^* X \times_{i^* i_* i^* j_* j^* X} i^* i_* i^* X \simeq i^* j_* j^* X \times_{i^* j_* j^* X} i^* X \simeq i^* X, \qquad \forall X \in \mathcal{X}.$$

Consequently, $i^*(\boxed{\mathfrak{R}})$ is a pullback square.

2. Case: j^*
Since $j^* j_* \simeq \mathrm{id}_{\mathcal{U}}$ and $j^* i_*$ is the functor constant at $1_{\mathcal{U}}$, the pullback simplifies to

$$j^* j_* j^* X \times_{j^* i_* i^* j_* j^* X} j^* i_* i^* X \simeq j^* X \times_{1_{\mathcal{X}}} 1_{\mathcal{X}} \simeq j^* X, \qquad \forall X \in \mathcal{X}.$$

Hence, $j^*(\boxed{\mathfrak{R}})$ and thus $\boxed{\mathfrak{R}}$ are both pullback squares.

□

B Finite Algebras of k-Affinoid Algebras

It is surprising and remarkable that the theory of finite complete modules over a **Noetherian** Banach ring A does not depend on the norm, the completeness and the bounded homomorphism from A. Instead such modules are determined solely by the algebraic part of their definition i.e. the fact that it is a finite A-module. This even carries over to algebras over Noetherian Banach k-algebras and thus affinoid algebras. The purpose of this section is to make these statements

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[86] In this case, i^* and j^* are automatically (Bousfield-)localizations by [Lan21, 5.1.23].

precise. We follow [BGR84, A3.7.2-4], supplemented with basic statements from Functional Analysis as covered in [Cla20], for example.

Notation B.0.1. For this subsection, A is Banach k-algebra unless stated otherwise. In contrast to the usual of (finite) A-Modules $A\text{-Mod}^{(\text{fin})}$ we denote by $A - \mathfrak{Mod}^{(\text{fin})}$ the category of completed normed A-modules with bounded (i.e. continuous) A-linear maps. Likewise, $A\text{-Ban}^{\text{fin}}$ denotes the category of Banach A-algebras which are finite A-modules together with bounded homomorphisms.

Definition B.0.2. Let A be a semi-normed ring. An element $a \in A$ is **topologically nilpotent** if $\lim_{n\to\infty} a^n = 0$. Denote the set of all topologically nilpotent elements by $\check{A}$.

Lemma B.0.3. *Let A be a semi-normed ring. Then, $\check{A}$ is a subgroup of $(A, +)$ and multiplicatively closed.*

Proof. Multiplicative closure is trivial. Clearly $0 \in \check{A}$. To show that $a - b \in \check{A}$ one uses a simple argument involing the sequences $(|a^n|)_n$ and $(|b^m|)_m$. See [BGR84, A.1.2.4/2]. ☐

Proposition B.0.4 ([BGR84, A.1.2.4/4])**.** *Let A be a complete semi-normed ring. Then, any element of the form $1 - y$, for $y \in \check{A}$, is a unit in A.*

Proof. Since A is complete it contains $\sum_{n=0}^{\infty} y^n$. Then, $(1 - y)(\sum_{n=0}^{\infty} y^n) = 1$. ☐

Lemma B.0.5 (Topological Nakayama Lemma [BGR84, A.1.2.4/6])**.** *Let A be a complete normed ring and let $M \in A\text{-Mod}$. Suppose there is a submodule $N \subset M$ and elements $x_1, ..., x_n \in M$ such that*

$$M \subset N + \oplus_{i=1}^{n} \check{A} x_i.$$

Then, $M = N$.

Proof. By assumption there are $c_{ij} \in \check{A}$ and $y_i \in N$, such that

$$x_i = y_i + \sum_{i=1}^{n} c_{ij} x_j$$

for every $0 \leq i \leq n$. Setting $x = (x_1, ..., x_n)^T$, $y = (y_1, ..., y_n)^T$ and $C = (c_{ij})_{i,j} \in \check{A}^{n\times n}$, yields $y = (I_n - C)x$, where $I_n \in A^{n\times n}$ is the unit matrix. By Lem. B.0.3 there is a $c \in \check{A}$ such that $\det(I - C) = 1 - c$. Thus, $\det(I - C)$ is invertible and therefore $x = (I_n - C)^{-1} y$. This means that $x_1, ..., x_n \in N$, hence $M = N$. ☐

Lemma B.0.6. *Let $M \in A\text{-Mod}^{\text{fin}}$ be equipped with an A- module norm $|-|$. If $\widehat{M}$ is in A-Mod, then $M \cong \widehat{M}$ as A-modules.*

Proof. ☐

Corollary B.0.7. *A $M \in A\text{-Mod}$ is Noetherian if and only if all its submodules are closed. Thus, A is Noetherian if and only if all its ideals are closed.*

Proof. By Baire's category Theorem (BCT) (see e.g. [Cla20, Thm.5.1]) ☐

Proposition B.0.8. *Any k-affinoid algebra is Noetherian.*

Proposition B.0.9. *1. Any $M \in A\text{-Mod}^{\text{fin}}$ admits a complete A-module norm, unique up to equivalence.*
2. For any $M, M' \in A - \mathfrak{Mod}^{fin}$ any (ordinary) homomorphism $\varphi \colon M \to M'$ of A-modules is bounded.
In consequence, the assignment of a norm as in 1. is an equivalence of categories

$$\Phi \colon A\text{-Mod}^{\text{fin}} \to A\text{-}\mathfrak{Mod}^{\text{fin}}$$

whose inverse is the forgetful functor.

Proof. It is convenient to prove *2.* before *1.*
2. Let $m_1, ..., m_n \in M$ be generators of M as A-module. Let $\pi \colon A^n \to M$ be the corresponding epimorphism. By Definition, π is bounded by the constant $\max_{i=1,...,n} |m_i|_M$. Likewise, $\varphi \circ \pi$ is bounded by $\max_{i=1,...,n} |\phi(m_i)|_M$. The Open Mapping Theorem [Cla20, II.5.5] implies that π is open, since it is surjective. In consequence, φ is continuous, thus bounded.

In particular *2.* implies that the norm on any $M \in A - \mathfrak{Mod}$ is unique up to equivalence. Indeed, for any two complete norms on M the identity id_M of A-Mod induces a bounded map $(M, |{-}|_1) \to (M, |{-}|_2)$ and vice versa.

1. Let $\pi \colon A^n \to M$ be a epimorphism exhibiting M as finite A module. Since A^n is Noetherian, $\ker(\pi)$ is closed by Cor. B.0.7. Therefore the residue norm on $M \cong A^n / \ker(\pi)$ makes M a complete, normed A-module. ☐

Proposition B.0.10. *Let A be a Noetherian Banach k-algebra and $A\text{-Alg}^{\text{fin}}$ the category of finite A-algebras. Then, the functor of Prop. B.0.9 extends to an equivalence*

$$\Phi \colon A\text{-Alg}^{\text{fin}} \to A\text{-Ban}^{\text{fin}}$$

whose inverse is the forgetful functor.

Proof. ☐

Lemma B.0.11 ([BGR84, 3.7.3/6])**.** *Let A and B be Noetherian Banach k-algebras and let $\varphi \colon A \to B$ a bounded homomorphism. The isomorphism $\Phi \colon \text{Mod}^{\text{fin}} \to \mathfrak{Mod}^{\text{fin}}$ sends*

1) $M \otimes_A N$ to $\Phi(M) \widehat{\otimes}_A \Phi(N) \in A\text{-}\mathfrak{Mod}^{\text{fin}}$ for all finite A-modules M, N;

2) $B \otimes_A M$ to $B \widehat{\otimes}_A \Phi(M) \in B\text{-}\mathfrak{Mod}^{\text{fin}}$ for all finite A-modules M;

3) $B \otimes_A C$ to $B \widehat{\otimes}_A \Phi(C) \in B\text{-Ban}^{\text{fin}}$ for all finite Banach A-algebras C.

In particular, the following diagram commutes

$$\begin{tikzcd}
A\text{-Alg}^{\text{fin}} \arrow[r, "\Phi"] \arrow[d, "B\otimes_A -"] & A\text{-Ban}^{\text{fin}} \arrow[d, "B\widehat{\otimes}-"] \\
B\text{-Alg}^{\text{fin}} \arrow[r, "\Phi"] & B\text{-Ban}^{\text{fin}}
\end{tikzcd}$$

Proof. Since the inverse of Φ is the forgetful functor, it suffices to show that the underlying sets in *1.-3.* coincide. The argument for all three assertions is exactly the same, so we restrict ourselves to only prove *1.* Moreover, we regard M and N as equipped with the unique norms that provide $\Phi(M)$ and $\Phi(N)$ with their Banach A-module structure.

If M is a free A-module, the assertions follow directly from Lem. 6.1.8 and the fact that $A\widehat{\otimes}_A N \cong N$ for all Banach A-modules N, which is explained in [BGR84, 2.1.7/6]. Otherwise, consider a free resolution

$$A^m \xrightarrow{\phi} A^n \longrightarrow M \longrightarrow 0.$$

Tensoring with N yields the exact sequence

$$A^m \otimes N \xrightarrow{\phi\otimes id_N} A^n \otimes N \xrightarrow{\psi\otimes id_N} M \otimes N \longrightarrow 0$$

ψ is admissible, as a morphism of finite Banach A- modules and moreover surjective. Then [BGR84, 2.1.8/6] shows that $\psi \otimes id_N$ is admissible. Since it is a map of finite Banach A-modules, $\phi \otimes id_N$ is admissible as well. Thus, apply Lem. 6.1.5 and deduce that the following sequence is exact as well

$$A^m\widehat{\otimes}N \xrightarrow{\phi\widehat{\otimes}id_N} A^n\widehat{\otimes}N \xrightarrow{\psi\widehat{\otimes}id_N} M\widehat{\otimes}N \longrightarrow 0$$

In conclusion, $M\widehat{\otimes}N$ is a finite A-module as well and thus $M\otimes N \xrightarrow{\widehat{(-)}} M\widehat{\otimes}N$ is an isomorphism by Lem. B.0.6. ☐

C Advanced Facts in Algebraic Geometry

C.1 Resolution of Singularities

Definition C.1.1 (Normal Crossing Divisor, [Kol07], [Sta25]). Let X be a smooth variety. An effective divisor[87] $E = \sum E^i$ is a **strict normal crossing divisor** on X, if each E^i is smooth, and for each point $x \in X$ one can choose local coordinates $z_1, \dots, z_n \in \mathfrak{m}_x$ in the maximal ideal of the local ring $\mathcal{O}_{X,x}$ such that for each i

1) either $x \notin E^i$, or

2) $E^i = (z_{c(i)} = 0)$ in a neighbourhood of x for some $c(i)$, and

3) $c(i) \neq c(i')$ if $i \neq i'$.

A subvariety $Z \subset X$ has **simple normal crossings** with E if one can choose $z_1, \dots, z_n$ as above such that in addition

4) $Z = (z_{j_1} = \cdots = z_{j_s} = 0)$ for some $j_1, \dots, j_s$, again in some open neighbourhood of x.

---

[87] Note that, Cartier

In particular, Z is smooth, and some of the E^i are allowed to contain Z. If E does not contain Z, then $E|_Z$ is again a simple normal crossing divisor on Z.

The famous Theorem of Hironaka [Hir64] shows, that every scheme X, which is of finite type over a field of characteristic zero, admits a projective morphism from a smooth scheme, which restricts to an isomorphism over the smooth points of X, Moreover, the inverse image of the singular points yields a strict normal crossing divisor. Kollár [Kol07] provides a proof of this Theorem, amongst an brilliant outline of the strategy and an overview of different available methods. Along the way, Kollár [Kol07] establishes the following functorial statement.

Proposition C.1.2 (Iterated Blowup Functor, [Kol07, 35])**.** *Consider a triple $(X, \mathcal{I}, E)$, where X is a smooth scheme of finite type over a field k of characteristic zero, $\mathcal{I} \subset \mathcal{O}_X$ is an ideal sheaf that is not zero on any irreducible component of X and E is a simple normal crossing divisor on X. Then, one can find a sequence of blow-ups*

$$\begin{tikzcd}
\mathcal{BP}(X,\mathcal{I},E) := \Pi : X_r \arrow[r, "\pi_{r-1}"] & X_{r-1} \arrow[r, "\pi_{r-2}"] & \dots \arrow[r, "\pi_1"] & X_1 \arrow[r, "\pi_0"] & X_0 = X, \\
& Z_{r-1} \arrow[u, phantom, "\cup"] & \dots & Z_1 \arrow[u, phantom, "\cup"] & Z_0 \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

all centers of blow-ups are smooth and have simple normal crossing with E.
Moreover, the assignment $\mathcal{BP}\colon (X, \mathcal{I}, E) \mapsto \mathcal{BP}(X, \mathcal{I}, E)$ satisfies the following properties:

1) *The pull-back $\Pi^*\mathcal{I} \subset \mathcal{O}_{X_r}$ is the ideal sheaf of a simple normal crossing divisor.*

2) $\Pi : X_r \to X$ *is an isomorphism over* $X \setminus \operatorname{supp}(\mathcal{O}_X/\mathcal{I})$.

3) *$\mathcal{BP}$ commutes with smooth morphisms, i.e. if $h\colon Y \to X$ is smooth, then pulling back $\mathcal{BP}(X, \mathcal{I}, E)$ along h (and deleting every blow-up $h^*\pi_i$, whose center is now empty, provides the sequence $\mathcal{BP}(Y, h^*\mathcal{I}, h^{-1}E)$.*

4) *$\mathcal{BP}$ commutes with change of fields i.e. if $k \subset L$ is a field extension, then pulling the sequence $\mathcal{BP}(X, \mathcal{I}, E)$ back along* $\operatorname{Spec}(L) \to \operatorname{Spec}(L)$ *yields the blow-up sequence $\mathcal{BP}(X_L, \mathcal{I}_L, E_L)$.*

5) *$\mathcal{BP}$ commutes with closed embeddings, whenever $E = \emptyset$. See [Kol07, p.24] for details.*

This Proposition then implies the Main Theorem of Hironaka, which Kollar phrases as follows.

Theorem C.1.3 (Resolution of Singularities, [Hir64], [Kol07, Thm.27])**.** *There is a blow-up sequence functor $\mathcal{BR}(X)$:*

$$\begin{tikzcd}
\mathcal{BR}(X) := \Pi : X_r \arrow[r, "\pi_{r-1}"] & X_{r-1} \arrow[r, "\pi_{r-2}"] & \cdots \arrow[r, "\pi_1"] & X_1 \arrow[r, "\pi_0"] & X_0 = X \\
& Z_{r-1} \arrow[u, phantom, "\cup"] & & Z_1 \arrow[u, phantom, "\cup"] & Z_0 \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

where $Z_i \subset X_i$ is the center of the blow-up $\pi_i : X_{i+1} \to X_i$. $\mathcal{BR}$ is defined on all schemes X of finite type over a field k of characteristic zero, satisfying the following conditions.

1) *X_r is smooth.*

2) $\Pi : X_r \to X$ *is an isomorphism over the smooth locus* X^{sm}.

3) $\Pi^{-1}(\operatorname{Sing} X)$ *is a divisor with simple normal crossings.*

4) $\mathcal{BR}$ *commutes with smooth morphisms and with change of fields*
i.e. if $h\colon Y \to X$ *is smooth, then pulling back* $\mathcal{BR}(X, \mathcal{I}, E)$ *along* h *(and deleting every blow-up* $h^*\pi_i$*, whose center is now empty, provides the sequence* $\mathcal{BR}(Y, h^*\mathcal{I}, h^{-1}E)$*. The same holds for every morphism* $i\colon \operatorname{Spec}(L) \to \operatorname{Spec}(k)$.

Sketch of Proof. First, assume X is affine and embed X into a smooth affine scheme A with $\dim A \geq \dim X + 2$. Define $\mathcal{I} \subset \mathcal{O}_A$ to be sheaf of ideals corresponding to the closure $\bar{X}$ of X in A. Then, apply Prop C.1.2 to obtain a sequence of blow-ups $\mathcal{BR}(A, \mathcal{I}, \emptyset)$ as above. Let $Z_j \subset P_j$ be the first center in this sequence, such that $g := \pi_0 \cdots \pi_{j-1}\colon Z_j \to \bar{X}$ is birational. Let F_{j+1} be the exceptional divisor of the blow-up $\pi_{j+1}\colon P_{j+1} \to P_j$ and let $F'_{j+1} \subset P_r$ be its birational transform. Then, F'_{j+1} is a smooth divisor, since it lies in the total exceptional divisor of the map Π. It can then be shown, that $\Pi|_{F'_{j+1}}$ is smooth over $\bar{X}^{sm}$. Since it factors as $\Pi|_{F'_{j+1}}\colon F'_{j+1} \to Z_j \xrightarrow{g} \bar{X}$, the map g has to be smooth over $\bar{X}^{sm}$. Since it is also birational, this shows that g is an isomorphism over $\bar{X}^{sm}$. Moreover, since Z_j is smooth, g cannot be an isomorphism over the singular points $\bar{X}^{sing}$ of $\bar{X}$. Therefore,

$$g^{-1}(\bar{X}^{sing}) = Z_j \cap \operatorname{Ex}_{\text{tot}}(\pi_0 \circ \ldots \circ \pi_{j-1})$$

where Ex_{tot} denotes the exceptional divisor of the iterated blowup. But [Kol07, 26] shows that the construction of blow-ups forces that Z_j has strict normal crossings with $\operatorname{Ex}_{\text{tot}}(\pi_0 \circ \ldots \circ \pi_{j-1})$ hence $g\colon Z_j \to \bar{X}$ is a resolution of singularities and therefore restricts to a resolution of singularities on X.One then continues by showing that this construction is independent of A, stable under bluings and functorial in general ☐

Remark C.1.4. The first general statement algebraic geometry in form of Prop. C.1 has been proved by Hironaka [Hir64] in 1964. Since then there have been many advances towards extending this Theorem to fields of arbitrary characteristic. Lately, Hironaka himself [Hir17] and Hu [Hu22] have announced (weak) versions of resolutions over fields of positive characteristic. However, the correctness of their proofs seems to be still under discussion (c.f. [htt]).

In view of Rmk. C.1.4, it seems reasonable to phrase results whose proof relies on a statement in the from of Prop. C.1 in a way that highlights this dependency and not the dependency on the characteristic zero. This will be done by using the convention below.

Definition C.1.5. A field k is said to admit (strong) resolutions of singularities, if the statement of Prop. C.1 is true for varieties over k.

Corollary C.1.6. *Given a pullback diagram of schemes*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\psi"] & U \arrow[d, "\varphi"] \\ Y \arrow[r, "g"] & V \end{tikzcd}$$

where φ *is a resolution of singularities, then* ψ *is a resolution of singularities as well.*

C.2 Compactification

Theorem C.2.1 (Nagata's Compactification Theorem, [Sta25, Tag 0F41])**.** *Let S be a qcqs scheme. Any separated and locally finite type morphism of schemes $f\colon X \to S$ admits a compactification i.e. a factorization*

$$\begin{tikzcd}
X \arrow[rr, "j"] \arrow[dr, "f"'] & & \overline{X} \arrow[dl, "\bar{f}"] \\
& S &
\end{tikzcd}$$

where j is an open immersion and $\bar{f}$ is proper.

Definition C.2.2. A morphism of schemes $f\colon X \to S$ is said to be **compactifyable**, if f admits a compactification in the sense of Thm. C.2.1.

Note. The converse is also true. Any morphism $X \to S$, that admits a compactification is separated and finite type, since both open immersions and proper morphisms are separated and finite type.

Corollary C.2.3 (Smooth Compactification)**.** *Assume, that k admits resolutions of singularities and let $X \in \mathrm{Sm}_{/k}$ be separated. Then, X admits smooth compactification, that is a dense open immersion $i\colon X \mathrel{-\!\!\circ\!\!\to} \tilde{X}$ into a proper smooth k-scheme $\tilde{X}$. Moreover, the complement $\tilde{X} \setminus i^{-1}X$ is a divisor with strict normal crossings.*

This has been proved originally in [Hir64, 0.§5.Cor 3], but seems not to appear in standard modern literature. Relying on Prop. C.1.2, this can be proved as follows.

Proof. By Thm. C.2.1 X admits a compactification $j\colon X \mathrel{-\!\!\circ\!\!\to} P$. Let $f\colon Y \to P$ be a resolution of singularities. Then, f induces an isomorphism $g : f^{-1}(P^{sm}) \to P^{sm}$, where P^{sm} are the smooth points of P. Since X is smooth, g^{-1} restricts to a dense open immersion

$$h\colon X \mathrel{-\!\!\circ\!\!\to} P^{sm} \overset{g^{-1}}{\cong} f^{-1}(P^{sm}) \subset Y.$$

Let $E := Y \setminus g^{-1}P^{\mathrm{sm}}$ be the normal crossing divisor corresponding to the inverse image of the singular points of P. Denote by $\mathcal{I} \subset \mathcal{O}_Y$ the sheaf of ideals corresponding to the closed subset $Z := f^{-1}(P^{sm} \setminus X)$. Applying Prop. C.1.2 yields a blow-up sequence $\mathcal{BP}(Y, \mathcal{I}, E)$. Let $\Pi\colon \tilde{X} \to Y$ be the outer terms of this sequence. Then, $\Pi^{-1}Z$ is a normal crossing divisor and Π is an isomorphism over $Y \setminus Z$. This in turn yields the anticipated dense open immersion

$$i\colon X \xrightarrow{h} Y \setminus Z \xrightarrow{\Pi^{-1}} \tilde{X}$$

since $\Pi^{-1}Z \sqcup E = \Pi^{-1}(Y \setminus h(X)) = \tilde{X} \setminus \Pi^{-1}h^{-1}(X)$ is strict normal crossing divisor. Moreover, $\Pi\colon \tilde{X} \to Y$ and $f\colon Y \to P$ are projective by construction and P is proper. Thus, $\tilde{X}$ is a proper k-scheme. ☐

The last line of this proof shows, that if the compactification is projective, then the smooth compactification remains projective.

Corollary C.2.4 (Smooth Projective Compactifiaction)**.** *Assume k admits a (strong) resolution of singularities and $X \in \mathrm{Sm}_{/k}$ admits a open dense open immersion into a projective scheme (e.g. X is a smooth affine or smooth quasi-projective variety). Then, there is a dense open immersion $j\colon X \hookrightarrow \tilde{X}$ into smooth projective k-scheme and the complement of $j(X)$ is a strict normal crossing divisor.*

Note. As shown in [Har77, II.Ex.7.13], the statement of Cor. C.2.4 is false, unless X is assumed to admit an open dense open immersion into a projective scheme.

Definition C.2.5 (Derived lower shriek)**.** Let $f\colon X \to Y$ be a morphism of schemes, which admits a compactification

$$\begin{tikzcd} X \arrow[rr, "j"] \arrow[dr, "f"'] & & \bar{X} \arrow[dl, "\bar{f}"] \\ & Y & \end{tikzcd}.$$

Define $Rf_! := R\bar{f}_* j_!$, where $j_!$ is the exact functor of [Sta25, Tag 03S2].

C.3 A Weak Version of Hartog's Lemma

The following Theorem allows to extend coherent sheaves over complements of codimension ≤ 2. Originally it is due to Kollár [Kol17]. As it allows for a precise characterisation of when the direct image is coherent, it has been covered in [Sta25, Tag 0BL8], where it appears in the phrasing below.

Proposition C.3.1 (Kollar,[Kol17], [Sta25, Tag 0BK3])**.** *Let $j : U \to X$ be an open immersion of locally Noetherian schemes with complement Z. Let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module. The following are equivalent:*

1) *$j_*\mathcal{F}$ is coherent;*
2) *for $x \in \mathrm{Ass}(\mathcal{F})$ and $z \in Z \cap \overline{\{x\}}$, and any associated prime $\mathfrak{p}$ of the completion $\widehat{\mathcal{O}}_{\{x\},z}$, we have*
$$\dim\left(\widehat{\mathcal{O}}_{\{x\},z}/\mathfrak{p}\right) \geq 2.$$

C.4 Étale Morphisms

Definition C.4.1. Let $f\colon A \to B$ be a morphism in CRing. Then,

- f is **formally étale** if for every square zero extension $R \to R_0$ and any commutative square, formed by the full arrows in the diagram below, there exists a **unique lift** σ

$$\begin{tikzcd} A \arrow[r] \arrow[d] & R \arrow[d] \\ B \arrow[r] \arrow[ur, dashed, "\sigma"] & R_0 \end{tikzcd}$$

  such that the whole diagram commutes.

- f is of **finite presentation**, if $B = {}^{A[X_1, \ldots, X_n]}\!/\!_{(f_1, \ldots, f_m)}$ for some $m, n \geq 0$ and polynomials $f_1, \ldots, f_m$.

- f is **étale** if it is of formally étale and of finite presentation.

Definition C.4.2. A morphism of schemes $f\colon X \to Y$ is formally étale/finitely presented/étale at a point $x \in X$ if there is are affine neighbourhoods $\operatorname{Spec}(A)$ of $f(x)$ and $\operatorname{Spec}(B) \subset f^{-1}(\operatorname{Spec}(A))$ of x, such that $A \to B$ is formally étale/of finite presentation/étale.

The next Lemma shows, that étale morphisms are "smooth of dimension 0".

Lemma C.4.3. *A morphism $A \to B$ is étale iff $B \cong \frac{A[x_1,\ldots,x_r]}{f_1,\ldots,f_r}$ for some $r \in \mathbb{N}$ and the Jacobian $\left(\frac{\partial f_j}{\partial x_i}\right)_{i,j}$ is invertible.*

Corollary C.4.4. *Étale morphisms are smooth, in particular unramified.*

Lemma C.4.5. *1) Étale morphisms of schemes are stable under base change.*

2) Composites of Étale morphisms of schemes are étale.

3) Étale morphisms have the cancellation property wrt. unramified morphisms i.e. if

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

where g and $g \circ f$ are étale, then f is étale as well.

Proof. *1)* and *2)* are similar to Lem. 6.2.1, *3)* is [GW23, 18.30]. □

Lemma C.4.6. *For any étale morphism $f\colon X \to S$ and any point $s \in S$, the fiber $X_s := X \times_S \kappa(s)$ is a (possibly infinite) coproduct $\coprod_{i \in I} \operatorname{Spec}(k_i)$ of finite separable field extensions $k_i/\kappa(s)$.*

Corollary C.4.7. *Étale morphisms into $X \in \mathrm{Sch}^{lft}_{/k}$ are open.*

Lemma C.4.8 ([Sta25, Lemma 04DH])**.** *Let $\pi : U \to X$ be finite étale a morphism of schemes. Let $x \in X$. Assume that*

1) $\pi^{-1}(\{x\}) = \{u\}$ for a $u \in U$

2) $\kappa(s) = \kappa(x)$

Then there exists an open neighbourhood V of x such that

$$\pi|_{\pi^{-1}(V)} : \pi^{-1}(V) \to V$$

is an isomorphism.

Example C.4.9. For a field k, each $X \in \operatorname{Spec}(k)_{ét}$ is a disjoint union of finite separable extensions of k.

Proposition C.4.10 ([Sta25, Tag 021E])**.** *The inclusion $(\mathrm{Aff}, \tau_{ét}) \hookrightarrow (\mathrm{Sch}, \tau_{ét})$ satisfies the conditions of Lem. 2.6.12. Thus it induces an equivalence*

$$\mathrm{Shv}_{ét}(\mathrm{Aff}) \simeq \mathrm{Shv}_{ét}(\mathrm{Sch})$$

Likewise, the inclusion $(X_{\mathrm{Aff}}, \tau_{\acute{e}t}) \hookrightarrow (X_{\acute{e}t}, \tau_{\acute{e}t})$ *induces an equivalence*

$$\mathrm{Shv}_{\acute{e}t}(X_{\mathrm{Aff}}) \simeq \mathrm{Shv}_{\acute{e}t}(X)$$

for every scheme X*.*

C.5 Local structure of the étale site

Definition C.5.1. A **geometric point** is a morphism $\bar{x}\colon \mathrm{Spec}(K) \to X$, where K is an algebraically closed field. If $x \in X$ is the image of $\bar{x}$, then x is said to be the underlying point of $\bar{x}$.

Observation C.5.2. Let $x \in X$ be a fixed point with residue field $\kappa(x)$ and let $K/\kappa(x)$ be an algebraic closure of $\kappa(x)$. The geometric points $\bar{x}$ of X, whose underlying point is x, correspond to the group $\mathrm{Aut}(K/\kappa(x)$ of automorphisms $\sigma\colon K \to K$ fixing $\kappa(x)$.

Observation C.5.3. By ... and ... pulling back along a geometric point $\bar{x}\colon \mathrm{Spec}(K) \to X$ yields the following point of the étale site

$$F_{\bar{x}} := \bar{x}^*\colon X_{\acute{e}t} \xrightarrow{\bar{x}\times_X -} \mathrm{Spec}(K)_{\acute{e}t} \xrightarrow{y} \mathrm{Set},\ U \mapsto |U_{\bar{x}}|\,.$$

where $|U_{\bar{x}}|$ is the underlying set of the fiber $U_{\bar{x}} := U \times_X \mathrm{Spec}(K)$.

Definition C.5.4. An **étale neighbourhood** of a geometric point $\bar{x}$ is a pair (U, u), where $U \to X \in X_{\acute{e}t}$ is an étale morphism and u is a lift of $\bar{x}$ to U i.e. a morphism such that the following diagram commutes

$$\begin{tikzcd}
 & U \arrow[d] \\
\mathrm{Spec}(K) \arrow[ur, "u"] \arrow[r, "\bar{x}"] & X
\end{tikzcd}$$

A morphism between two étale neighbourhoods (U, u) and (V, v) of $\bar{x}$ is a morphism $f\colon U \to V$ in $X_{\acute{e}t}$, which induces a commutative diagram

$$\begin{tikzcd}
 & U \arrow[d, "f"] \\
\mathrm{Spec}(K) \arrow[ur, "u"] \arrow[r, "v"] & V
\end{tikzcd}$$

Let $\mathrm{Nb}_{\acute{e}t}(\bar{x})$ denote the category of étale neighbourhoods of a geometric point $\bar{x}$.

Lemma C.5.5. *Let* $f : X \to S$ *be a morphism of schemes. Let* $x_1, \ldots, x_n \in X$ *be points having the same image* s *in* S*. Assume* f *is étale at each* x_i*. Then there exists an étale neighbourhood* $(U, u) \to (S, s)$ *and opens* $V_{i,j} \subset X_U$*,* $i = 1, \ldots, n$*,* $j = 1, \ldots, m_i$ *such that*

1) $V_{i,j} \to U$ *is an isomorphism,*

2) u *is not in the image of* $V_{i,j} \cap V_{i',j'}$ *unless* $i = i'$ *and* $j = j'$*, and*

3) *any point of* $(X_U)_u$ *mapping to* x_i *is in some* $V_{i,j}$*.*

Lemma C.5.6. *Let $f : X \to S$ be an étale morphism, $s \in S$ and $f^{-1}(s) = \{x_1, ..., x_n\}$. Then there exists an étale neighbourhood $(U, u) \to (S, s)$ and a finite disjoint union decomposition*

$$X_U = W \amalg \coprod\nolimits_{i,j} V_{i,j}$$

of schemes such that

1) *$V_{i,j} \to U$ is an isomorphism,*

2) *the fibre W_u contains no point mapping to any x_i.*

In particular, if $f^{-1}(\{s\}) = \{x_1, \ldots, x_n\}$, then the fibre W_u is empty.

C.6 Base Change

Theorem C.6.1 ([Sta25, Tag 0A48])**.** *Let A be a henselian local ring. Let X be a proper scheme over A with closed fibre X_0. Then the functor*

$$\mathrm{F\acute{E}t}(X) \to \mathrm{F\acute{E}t}(X_0), \quad U \longmapsto U_0 = U \times_X X_0$$

is an equivalence of categories.

Corollary C.6.2. *Let K/k be an extension of algebraically closed fields. Let X be a proper scheme over* $\mathrm{Spec}(k)$*. Let $X_K := X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K)$. Then, pulling back induces an equivalence of categories.*

$$\mathrm{F\acute{E}t}(X) \to \mathrm{F\acute{E}t}(X_K), \quad U \longmapsto U_K$$

C.7 Proper Base Change

Theorem C.7.1 (Abelian Proper Base Change)**.** *For every pullback diagram of schemes*

$$\begin{tikzcd}
X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
Y' \arrow[r, "f'"] & Y
\end{tikzcd}$$

where f is a proper morphism, the push-pull transformation

$$g^* R^+ f_* \to R^+ f'_* g'^*$$

is an equivalence in $\mathrm{Fun}(\mathrm{D}^+_{\mathrm{tor}}(X), \mathrm{D}^+(Y'))$ *i.e. when restricted to the derived category of bounded below chain complexes of abelian sheaves on X with finite stalks.*

In order to establish Thm. C.7.1, it suffices to prove the classical statement of [AGVb, XII.Thm.5.1]. This means we can reduce to prove

Theorem C.7.2. *For every pullback diagram of schemes as in Thm. C.7.1 with f proper and an abelian sheaf $F \in$* $\mathrm{Shv}(X; \mathrm{Ab})$ *with finite stalks, the push-pull transformation induces an isomorphism in* $\mathrm{D}^+(Y')$

$$g^* R^* f_* F \simeq R^* f'_* g'^* F$$

where R^*f_* and $R^*f'_*$ are the derived higher image functors.

Proof of the reduction of Thm. C.7.1 to Thm. C.7.2. Suppose we have proved Thm. C.7.2. Let $F^\bullet \in \mathrm{D}^+_{\mathrm{tor}}(X)$ be a complex of étale abelian sheaves on X with finite stalks. Recall from [Wei94, Cor.10.5.7] that hyper-derived functors compute the cohomology groups of the induced functors on derived categories i.e $H^nR^+f_* \cong \mathbf{R}^n(f_*)$ and $H^nR^+f'_* \cong \mathbf{R}^n(f'_*)$, where $\mathbf{R}$ stands for the right hyper-derived functors. Let $I^{\bullet,\bullet}$ be a Cartan-Eilenberg resolution of $F^{\bullet,}$, then there are two converging spectral sequences by [Wei94, Prop.5.7.6],

$$\begin{aligned} &1.E_1^{p,q} = H^q(g^*f_*I^{p,\bullet}) \Rightarrow \mathbf{R}^{p+q}(g^*f_*F^\bullet) \\ &2.E_1^{p,q} = H^q(f'_*g'^*I^{p,\bullet}) \Rightarrow \mathbf{R}^{p+q}(f'_*g'^*F^\bullet) \end{aligned}$$

However, the E_1-pages of the spectral sequences coincide, as we have

$$H^q(g^*f_*I^{p,\bullet}) \cong g^*H^q(f_*I^{p,\bullet}) \cong g^*R^qf_*F^p \cong R^qf'_*g'^*F^p \cong H^q(f'_*g'^*I^{p,\bullet})$$

since g^* and g'^* are exact and $I^{p,\bullet}$ is an injective resolution of F^p. Convergence of the spectral sequences then yields

$$g^*\mathbf{R}^n(f_*F^\bullet) \cong \mathbf{R}^n(g^*f_*F^\bullet) \cong \mathbf{R}^n(f'_*g'^*F^\bullet) \cong \mathbf{R}^n(f'_*)g'^*F^\bullet \text{ for all } n \geq 0$$

In consequence, $g^*R^+f_*F^\bullet \simeq (R^+f'_*)g'^*F^\bullet$ in $\mathrm{D}^+(Y')$ for every $F^\bullet \in \mathrm{D}^+_{\mathrm{tor}}(X)$. ☐

C.8 Galois Categories

As the Terminology varies throughout the literature, we provide a explicit Definition of a Galois category.

Definition C.8.1 ([BS14, 7.2.5])**.** Let $\mathcal{C}$ be a category and let $F\colon \mathcal{C} \to \mathrm{Sets}$ be a functor. The pair $(\mathcal{C}, F)$ is called a *Galois category* if the following conditions hold:

1) $\mathcal{C}$ has finite limits and finite colimits;
2) every object of $\mathcal{C}$ is a finite (possibly empty) coproduct of connected objects;
3) $F(X)$ is finite for all $X \in \mathrm{Ob}(\mathcal{C})$; and
4) F reflects isomorphisms and is exact.

Here an object $X \in \mathcal{C}$ is said to be *connected* if it is not initial and for every monomorphism $Y \to X$, either Y is initial or $Y \to X$ is an isomorphism.
An object $X \in \mathcal{C}$ is a Galois object, if

Lemma C.8.2. *Any connected objected V in a Galois category $\mathcal{C}$ admits a morphism $U \to V$ from a Galois object.*

D A Note on Classical Descent Data

In classical algebraic geometry, one often encounters so called "descent datums", which appear rather complicated and provide some trouble to work with. In this section we want to relate the classical notion to the higher categorical viewpoint, on which we rely in the proof of the Riemann existence Theorem 9.1.9. Descent data – as in [SGA1, IX.§3] or [Sta25, Tag 0BTH] – are always defined relative to a morphism $f\colon X \to S$. Classically a descent datum consists of a scheme $V \to X$ and an isomorphism $\phi\colon V \times_S X \to X \times_S V$ over $X \times_S X$, which satisfies a cocycle condition.

Definition D.0.1. Let $\mathcal{C}$ be a ∞-category with pullbacks. Let $f\colon X \to S$ be a morphism in $\mathcal{C}$ and $\check{\mathrm{C}}(f) := \check{\mathrm{C}}(f\colon X \to S)_\bullet \in \mathcal{C}^{\Delta^{\mathrm{op}}}$ its Čech nerve. A descent datum is simplicial object $D_\bullet \in \mathcal{C}^{\Delta^{\mathrm{op}}}$ is which admits a cartesian transformation[88] $D_\bullet \to \check{\mathrm{C}}(f)$. The category of descent data over f is defined as

$$\mathrm{DescD}(f) := \mathrm{Fun}^{\mathrm{cart}}(\Delta^1, \mathcal{C}^{\Delta}) \times_{\mathcal{C}^{\Delta}} \{\check{\mathrm{C}}(f)_\bullet\}.$$

This means a descent datum is a diagram of the form

$$\begin{tikzcd}
\cdots & \cdots \\
D_4 \arrow[r] \arrow[d, shift left=3] \arrow[d, shift left=1] \arrow[d, shift right=1] \arrow[d, shift right=3] & X \times_S X \times_S X \times_S X \arrow[d, shift left=3] \arrow[d, shift left=1] \arrow[d, shift right=1] \arrow[d, shift right=3] \\
D_3 \arrow[r] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & X \times_S X \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] \\
D_2 \arrow[r] \arrow[d, shift left=1] \arrow[d, shift right=1] & X \times_S X \arrow[d, shift left=1] \arrow[d, shift right=1] \\
D_0 \arrow[r] & X
\end{tikzcd}$$

where all squares are pullback diagrams.

Observation D.0.2. Every descent datum is a groupoid. The motivation for the name 'descent datum' stems from the Rezk Characterisation 2.4.4 for ∞-topoi. If $\mathcal{C}$ is an ∞-topos and $Y' \to Y$ is a morphism fitting into diagram of augmented simplicial objects

$$\begin{tikzcd}
D_\bullet \arrow[r] \arrow[d] & \check{\mathrm{C}}(f)_\bullet \arrow[d] \\
Y' \arrow[r] & Y
\end{tikzcd}$$

then, $X \to Y$ is an effective epimorphism if and only if $D_\bullet \to Y'$ is.

Observation D.0.3. Suppose for simplicity, that $\mathcal{C}$ is 1-category. Let $f\colon X \to S$ be a morphism in $\mathcal{C}$. Then, pulling back the (augmented) Čech nerve $\check{\mathrm{C}}(f) \to S$ provides a functor

$$D_f\colon \mathcal{C}_{/S} \to \mathrm{DescD}(f), \qquad (g\colon E \to S) \mapsto g^* \check{\mathrm{C}}(f) \to \check{\mathrm{C}}(f)$$

---

[88] c.f. Thm. 2.4.4

where $g^* \check{\mathrm{C}}(f)$ denotes the groupoid

$$\begin{tikzcd}
\cdots \arrow[r, shift left=4.5] \arrow[r, shift left=1.5] \arrow[r, shift right=1.5] \arrow[r, shift right=4.5] & E \times_S X \times_S X \times_S X \arrow[r, shift left=3] \arrow[r] \arrow[r, shift right=3] & E \times_S X \times_S X \arrow[r, shift left=1.5] \arrow[r, shift right=1.5] & E \times_S X
\end{tikzcd}$$

The goal of this section is generalize [SGA1, IX.3.1], or equivalently [Sta25, Tag 0BTK] and show that this functor is fully faithful in case $\mathcal{C}$ is the small site of an object in a category similar to schemes or analytic space equipped with a suitable topology. Faithfulness is simple and persists already at the level of topological spaces.

Lemma D.0.4 ([Sta25, Tag 0BTJ])**.** *If $f\colon X \to S$ is a surjective map of topological spaces, then the functor of Obs. D.0.3*

$$D_f\colon \mathrm{Top}_{/S} \to \mathrm{DescD}(f)$$

is faithful.

Proof. Let $g\colon U \to S$, $h\colon V \to S$ be maps and $a, b \in \mathrm{Hom}_{\mathrm{Top}_{/S}}(U, V)$, such that their pullbacks

$$U \times_S X \underset{b \times_S id_X}{\overset{a \times_S id_X}{\rightrightarrows}} V \times_S X.$$

coincide. Then, for an $u \in U$ there is an $x \in X$, such that $ga(u) = gb(u) = f(u)$. Thus, there is an element $(a(u), x) = (a \times_S id_X)(u, x) = (b \times_S id_X)(u, x) = (b(u), x) \in V \times_S X$ mapping simultaneously to $a(u)$ and $(b(u))$ via pr_V, i.e $a(u) = b(u)$. ☐

Fullness is more subtle and requires a more specific situation, as found at the étale site of schemes or analytic spaces.

Convention D.0.5. Let C be a category with finite limits, which is a subcategory of Top and let τ denote a class of morphisms in C, which will we call admissible, such that

1) All objects in C are locally separated i.e. for every $X \in C$ and every point $x \in C$ there is an open neighbourhood $U \subset X$ of in x such that the diagonal $U \to U \times U$ is a closed immersion i.e. a homeomorphism onto a closed subset. Notice, that all separated morphisms (i.e. those where the diagonal is a closed immersion) satisfy the cancellation property by the argument presented in [Sta25, Tag 01KV].
2) Admissible morphisms are stable under pullback and composition.
3) Sections[89] of admissible morphisms have open image.
4) Open immersions are admissible.
5) Any admissible morphism, which is universally bijective is an isomorphism.

Example D.0.6. The requirements listed in the convention above are by no means artificial. For example take $C = \mathrm{Sch}$. Then, C together with the collection open immersions clearly satisfies 1)-4), since all objects in C are locally affine, hence separated (c.f.[Sta25, Tag 01S7]). Likewise, taking the étale maps as admissible morphisms morphisms results in data satisfying the requirements above in particular because

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[89] i.e. morphisms s such that $f \circ s = id$ for an admissible morphism f.

3) Étale morphisms are open by Cor. C.4.7.

5) Étale morphisms, which are are universally bijective, are already isomorphisms (c.f.[Sta25, Tag 025G]).

Furthermore, the conditions 1)-4) are also met, when setting $C = \mathrm{Ber}_k$ and τ being the étale morphisms by Cor. 6.2.2 and the proof of [Sta25, Tag 025G].

Definition D.0.7. A map $f\colon X \to S$ of topological spaces is submersive if it is surjective and S carries the quotient topology with respect to f.

Lemma D.0.8. *Let C and τ be a category with admissible morphisms as in Conv. D.0.5. Let $f\colon X \to S$ be a universally submersive morphisms in C. Then, the functor of Obs. D.0.3*

$$D_f\colon C_{\tau/S} \to \mathrm{DescD}(f)$$

is fully faithful. Here, $C_{\tau/S} \subset C_{/S}$ is the full subcategory of admissible morphisms with target S.

The proof is essentially that of [Sta25, Tag 0BTK]

Proof. The functor D_f is faithful by Lem. D.0.4. To prove that is it is full, consider two admissible morphisms $u\colon U \to S$ and $v\colon V \to S$. Let $\alpha\colon u^*\,\check{\mathrm{C}}(f) \to v^*\,\check{\mathrm{C}}(f)$ be a morphism of descent data. In particular, α yields a morphism $a\colon U \times_S X \to V \times_S X$. The goal is to show that there is a morphism $U \to V$ over S, which pulls back to a.
First, observe, that for every open subset $V' \subset V$ setting $U' := \mathrm{pr}_U(a^{-1}(\mathrm{pr}_V^{-1}(V)))$ provides an open subset of U, since both $\mathrm{pr}_U\colon U \times_S X \to U$ and $\mathrm{pr}_V\colon V \times_S X \to V$ are submersive. Thus, it suffices to show the statement locally on V and by 1) V can be assumed to be a separated object of C, which by the cancellation property implies that $V \to S$ is separated. Let

$$\Gamma_a \subset U \times_S X \times_X V \times_S X \cong U \times_S V \times_X$$

be the graph, i.e. the image of $U \times_S X$ under the morphism

$$(id, a)\colon U \times_S X \to U \times_S X \times_X V \times_S X.$$

Note that this is a closed immersions, since it arises as pullback of the diagonal $X \times_S V \nrightarrow X \times_S V \times_X V \times_S X$ along (a, id), which itself is a closed immersion since V is separated. Furthermore, $\Gamma_a \subset U \times_S V \times_S X$ is an open subset by 3), since the morphism (id, a) defining Γ_a is a section to the admissible[90] morphism

$$\mathrm{pr}_{U\times_S X}\colon U \times_S X \times_X V \times_S X \to U \times_S X.$$

Since $\mathrm{pr}_{U\times_S V}\colon U \times_S V \times_S X \to U \times_S V$ is submersive, the image $\gamma := \mathrm{pr}_{U\times_S V}(\Gamma_a)$ is an open and closed subset of $U \times_S V$. In particular, the composition $g\colon \gamma \subset U \times_S V \xrightarrow{\mathrm{pr}_U} U$ is admissible by 2). By construction, the pullback of γ along $X \to S$ is Γ_a, which is the image of (id, a)

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[90] This morphism is obtained by pulling back the admissible morphism $U \to S$

and (id, a) is a section of $\mathrm{pr}_{X\times U}$. Hence, left upper horizontal arrow in the diagram of pullback squares below is an isomorphism

$$\begin{tikzcd}
\Gamma_a \arrow[r, "\cong"] \arrow[d] & X \times_S U \arrow[r] \arrow[d] & X \arrow[d] \\
\gamma \arrow[r] & U \arrow[r] & S
\end{tikzcd}$$

This makes $\gamma \to U$ an bijection, since all vertical maps are submersions. In particular, for any morphism $W \to U$, the pulled back map $\gamma \times_U W \to W$ is a bijection, since the vertical maps in the following pullback square are submersions and the upper horizontal map is an isomorphism

$$\begin{tikzcd}
\Gamma_a \times_U W \arrow[r] \arrow[d] & W \times_S X \arrow[d] \\
\gamma \times_U W \arrow[r] & W.
\end{tikzcd}$$

Hence, $\gamma \to U$ is an admissible and universally bijective morphism, hence an isomorphism by 5). Thus, $U \to \gamma \xrightarrow{\mathrm{pr}_V} V$ yields the desired morphism from U to V. □

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