Talk IV: Straightening-Unstraightening

Seminar Homotopy Theory - WS 2024/25
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Context: I originally prepared these notes as a support for my 180-min lecture in the seminar Homotopy Theory at LMU. The prerequisites for this talk are Chapters 1 and 2 of Markus Land's book [Lan21], as well as Section 3.1 on cocartesian fibrations. As mentioned in the title, the lecture aims to address the Straightening-Unstraightening Equivalence. Rather than outlining a technical proof, I opted for a more pedagogical approach. The focus is placed on the construction of the straightening and unstraightening functors at object level, providing an intuitive understanding of their behavior. Most of the time, I will closely follow [Lan21][§3.2-3.3], as this was also the main ressource for the seminar last semester. However, I also managed to link this talk to a recent Lemma, we accountered in the Topology 3 Course[1]. Since this seminar is intended for both Bachelor's and Master's students, I have worked out most arguments in considerable detail. I kindly ask the advanced reader for their understanding in this regard. As always, any suggestions for improvement are highly welcome. Please send them to `leon.schropp@t-online.de`.

1 Introduction

Definition 1.1
Let $\mathcal{B}$ be an ∞-category. We define (Co)Cart($\mathcal{B}$) $\subset \mathrm{Cat}_\infty/\mathcal{B}$ to be the (non-full) subcategory spanned by (co)cartesian fibrations over $\mathcal{B}$ and morphisms of (co)cartesian fibrations i.e.

commuting diagrams
$$\begin{tikzcd} \mathcal{E} \arrow[r, "f"] \arrow[dr, "p"'] & \mathcal{E}' \arrow[d, "p'"] \\ & \mathcal{B} \end{tikzcd}$$
s.t. f sends p-cocartesian morphisms to p'-cocartesian morphisms.

Let LFib($\mathcal{B}$), RFib($\mathcal{B}$) $\subset \mathrm{Cat}_\infty/\mathcal{B}$ the full subcategories of left, resp. right fibrations.

In the last talk we have proved that LFib($\mathcal{B}$)$\subset$ CoCart($\mathcal{B}$) is a full subcategory.

The topic of today's talk is the following statement, which has been one of the many, great achievements of Jacob Lurie:

Theorem 1.2 (Straightening-Unstraightening)
Let $\mathcal{B}$ be an ∞-category, then there are functors Str *and* Un*, called Straighenting and Unstraightening forming the following equivalences:*

$$CoCart(\mathcal{B}) \underset{Un}{\overset{Str}{\rightleftarrows}} \mathrm{Fun}(\mathcal{B}, \mathrm{Cat}_\infty)$$

$$Cart(\mathcal{B}) \underset{Un}{\overset{Str}{\rightleftarrows}} \mathrm{Fun}(\mathcal{B}^{op}, \mathrm{Cat}_\infty)$$

[1]see `https://www.mathematik.uni-muenchen.de/~land/Topo3.php`

This Theorem is deeply woven into the fabric of the Theory of ∞-Categories and is a key ingrediant to numerous proofs. The proof of the Yoneda Lemma for example, which we will come across next talk, will highly rely on this equivalence. Despite its importance, we will not be able to prove it during this talk. Usually one obtains this statement from the existance of a specific Quillen-equivalence. Due to the lengthy and rather technical nature of the proof of this Quillen-equivalence the only reference presenting a complete proof is Lurie's HTT [Lur09]. The goal of today's talk is therefore not to reach for a precise proof, but rather to get a feel for what these Str and Un functors do on object level. If the time allows, we might also be able to deduce some handy Corollaries and see how we can put them to work to obtain some precise topolocial results, that Markus presented the other week in his Topology 3 Course[2] . Before we will be able to come to this fun bit of the talk, however, we will have to dive into some definitons, facts and technical lemmas about marked simplicial sets.

2 Marked Simplicial Sets

Definitions 2.1
-Let $\mathbf{sSet}^+$ denote the 1-Category of marked simplicial Sets with objects given by pairs (X, M) with $X \in \mathbf{sSet}$ and $M \subset X_1$ specifying a subset of "marked" edges. A morphims $f : (X, M) \to (Y, N)$ in $\mathbf{sSet}^+$ is then given by a $f : X \to Y$ of $\mathbf{sSet}$, sending marked edges to marked edges, i.e. $f(M) \subset N$.

-Let $(\bullet)^\flat$, $(\bullet)^\sharp : \mathbf{sSet} \to \mathbf{sSet}^+$ be the functors given by marking all degenerate edges, resp. all edges of the input simplicial Set.

-Let $(\bullet)_\flat : \mathbf{sSet}^+ \to \mathbf{sSet}$ be the obvious forgetful functor.

-Given a cocartesian fibration $p : X \to S$, define a marked simplicial Set $X^\natural$ by marking all p-cocartesian edges of X. Doing so, we clearly obtian a morphism $p : X^\natural \to S^\sharp$.

Facts 2.2
1. The functors defined above yield the following adjunctions [HHR21]

$$(\bullet)^\flat \dashv (\bullet)_\flat \dashv (\bullet)^\sharp$$

2. $\mathbf{sSet}^+$ is cartesian closed [Lur09].

Definitions 2.3
-Let $K, L \in \mathbf{sSet}^+$. Define $\mathrm{Map}^\flat(K, L) := (L^K)_\flat \in \mathbf{sSet}$. Consequently, $\mathbf{sSet}^+$ is a simplicial 1-Category and by the same procedure as with Cat_∞ , we obtain a ∞-Category for $\mathbf{sSet}^+$.
To simplify notation, we'll omit the forgetful functor in any Functorspace, i.e. denote $\mathrm{Fun}((K)_\flat), \mathcal{E})$ simply by $\mathrm{Fun}(K, \mathcal{E})$.

-Additionally, let $p : \mathcal{E} \to \mathcal{B}$ be a cocartesian fibration.
Define $\mathrm{Fun}^{mcc}(K, \mathcal{E}) := \mathrm{Map}^\flat(K, \mathcal{E}^\natural) = ((\mathcal{E})^k)_\flat$. Clearly, this is the full subcategory of $\mathrm{Fun}(K, \mathcal{E})$ spanned by functors sending all marked morphisms of K to p-cocartesian morphisms of $\mathcal{E}$. The "mcc" in the superscript shall remind us of this restriction.

-Given a map $f : K \to \mathcal{B}^\sharp$, we write $\mathrm{Fun}^{mcc}_f(K, \mathcal{E})$ for the pullback in the following diagram:

$$\begin{tikzcd}
\mathrm{Fun}^{mcc}_f(K, \mathcal{E}) \arrow[r] \arrow[d] & \mathrm{Fun}^{mcc}(K, \mathcal{E}) \arrow[d, "p"] \\
\Delta^0 \arrow[r, "f"] & \$\mathrm{Fun}(K, \mathcal{B})
\end{tikzcd}$$

[2]see `https://www.mathematik.uni-muenchen.de/~land/Topo3.php`

NB: We can parametrize all p-cocartesian lifts of f with this pullback! Given an edge $f : \Delta^1 \to \mathcal{B} \in \mathbf{sSet}$, we get a marked edge $f : (\Delta^1)^\sharp \to \mathcal{B}^\sharp$, by marking all edges of $\mathcal{B}$. Then, $\mathrm{Fun}_f^{mcc}(K, \mathcal{E})$ are exactly the p-cocartesian lifts of f.

-Call a map of $\mathbf{sSet}^+$ marked left-anodyne if and only if it is contained in the saturated closure of maps of the following form [HHR21]:
1. $(A^\sharp \to B^\sharp) \boxtimes C \to D$, where $A \to B$ is left-anodyne, $\boxtimes$ the "Boxtensor" or "Boxproduct" and $C \to D$ a marked monomorphism, i.e. a monomorphism in **sSet**after applying $(\bullet)_\flat$.
2. $A^\flat \to B^\flat$, where $A \to B$ is left-anodyne.
3. $\mathrm{J}^\flat \to \mathrm{J}^\sharp$, with J being the Joyal-groupoid, given by a the nerve of the 1-Category containing only two objects and a morphims with an inverse betwen them.

Remark 2.4
I experienced that, one has to be a bit careful with the conventions here. In more recent literature e.g. [HHR21], the marked left-anodyne maps above are called cocartesian anodynes, whilst the maps listed under 1. are called left-anodyne. The upcoming fact shows why "cocartesian" appears in this naming convention. I decided to stick with the more traditional notation, as found in [Lan21] or [Lur09].

Note also, that one can give very explicit sets of generators for the class of marked left-anodynes (see [Lan21]), which makes problems involving them much more solveable. Although they are not overly complicated, will not pursue such arguments in this talk, due to their rather lengthy nature.

Fact 2.5 (Marked left-anodynes specify cocartesian Fibrations)
Any map $p : X \to S^\sharp$ of $\mathbf{sSet}^+$ has the right-lifting-property (**rlp**) wrt. all marked left-anodyne maps if and only if there exists a cocartesian fibration $p : Y \to S$ in **sSet**such that $X = Y^\natural$.

Fact 2.6
Any Boxproduct of marked left-anodynes with marked monomorphisms is again marked left-anodyne.

Note: This fact is a good example, where generators come in handy. If you like, you can find them in Markus book [Lan21] and prove this statement yourself (or look up the proof in [Lan21]).

Now, that we have endured the most tedious part of today's talk, let's see how we can put the newly learned definitions to work and deduce a statement, which will turn out useful, when we get to the construction of the **Str** functor.

Proposition 2.7
Let $p : \mathcal{E} \to \mathcal{B}$ a cocartesian fibration, $i : K \to L$ marked left-anodyne and $f : L \to B^\sharp$ a map of $\boldsymbol{sSet}^+$. Then $\mathrm{Fun}_f^{mcc}(L, \mathcal{E}) \xrightarrow{i_*} \mathrm{Fun}_{fi}^{mcc}(L, \mathcal{E})$ *is a trivial fibration.*

<u>Proof:</u> Let $j : S \to T$ be a mono of **sSet**. We want to solve the following lifting problem :

$$\begin{tikzcd} S \arrow[r] \arrow[d] & \mathrm{Fun}_{fi}^{mcc}(L, \mathcal{E}) \arrow[d] \\ T \arrow[r] \arrow[ur, dashed, "?"] & \mathrm{Fun}_{fi}^{mcc}(K, \mathcal{E}) \end{tikzcd}$$

By Definition of $\mathrm{Fun}_{fi}^{mcc}(L, \mathcal{E})$ and $\mathrm{Fun}_{fi}^{mcc}(K, \mathcal{E})$ as a pullbacks, we obtain that this lifting problem is equivalent to following lifting problem.

$$\Leftrightarrow \begin{tikzcd} S \arrow[r] \arrow[d] & \mathrm{Fun}^{mcc}(L,\mathcal{E}) \arrow[d] \\ T \arrow[r] \arrow[ur, dashed, "?"] & \mathrm{Fun}(L,\mathcal{B}) \times_{\mathrm{Fun}(K,\mathcal{B})} \mathrm{Fun}^{mcc}(K,\mathcal{E}) \end{tikzcd}$$

Using the adjunctions given in Fact 2.2, we can convert this lifting problem into the following one down to the left, using that $\mathrm{Fun}^{mcc}(L,\mathcal{E}) = ((\mathcal{E}^{\natural})^K)_{\flat}$. The adjunction of the mapping object yields that this again can be transformed into the lifting problem on the right, just as we have seen in Markus' talk (or see [Lan21]).

$$\Leftrightarrow \begin{tikzcd} S^{\flat} \arrow[r] \arrow[d] & (\mathcal{E}^{\natural})^K \arrow[d] \\ T^{\flat} \arrow[r] \arrow[ur, dashed, "?"] & (\mathcal{B}^{\sharp})^L \times_{(\mathcal{B})^K} (\mathcal{E}^{\natural})^K \end{tikzcd} \Leftrightarrow \begin{tikzcd} S^{\flat} \times L \sqcup T^{\flat} \times K \arrow[r] \arrow[d, "j \boxtimes i"] & \mathcal{E}^{\natural} \arrow[d] \\ T^{\flat} \times L \arrow[r] \arrow[ur, dashed, "?"] & \mathcal{B}^{\sharp} \end{tikzcd}$$

Examining the diagram to the right, the two previous Facts state precisely that this lifting problem can be solved, as j is a marked mono and i is marked left anoyne! ☐

Note 2.8. Observe, that giving a $(\Delta^0)^{\sharp} \to \mathcal{E}^{\natural}$ is the same datum, as giving a map $\Delta^0 \to \mathcal{E}$, since Δ^0 has only degenerate edges, which get mapped to degenerate ones anyways.
Hence, $\mathrm{Fun}^{mcc}((\Delta^0)^{\sharp},\mathcal{E}) \cong \mathrm{Fun}(\Delta^0,\mathcal{E}) \cong \mathcal{E}$. Now, let $b : \Delta^0 \to \mathcal{B}$ be a fixed element in $\mathcal{B}$, then the following diagram is a pullback by definition:

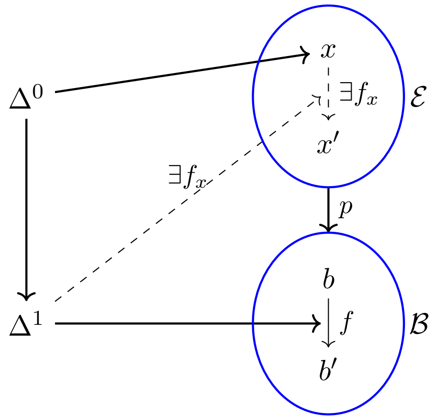
$$\begin{tikzcd} \mathrm{Fun}^{mcc}_b((\Delta^0)^{\sharp},\mathcal{E}) \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ \Delta^0 \arrow[r, "b"] & \mathrm{Fun}(\Delta^0,\mathcal{B}) \cong \mathcal{B} \end{tikzcd}$$

Hence, $\mathrm{Fun}^{mcc}_b((\Delta^0)^{\sharp},\mathcal{E})$ is just the fiber $\mathcal{E}_b$.
Furthermore, let $f : \Delta^1 \to \mathcal{B}$, say given by $b \xrightarrow{f} b'$. Then the Propositon above yields that the map $ev_0 : \mathrm{Fun}^{mcc}_f((\Delta^1)^{\sharp},\mathcal{E}) \to \mathcal{E}_b$, which is induced by the usual evaluation at 0, is a trivial fibrations, since $\{0\}^{\sharp} \to (\Delta^1)^{\sharp}$ clearly is marked left-anodyne.

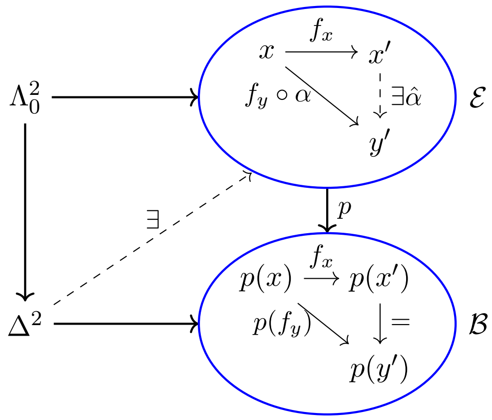
3 Straightening and Unstraightening on Object-Level

Finally, we can begin with the promised motivation for **Str** and **Un**. As mentioned earlier, providing a complete proof would exceed the scope of this talk, by far. A complete construction of **Str** on the object level can be found in [Lan21], where it fills a whole section of this book. Let's start some informal thoughts, what these functors **Str** and **Un** should do, instead.

<u>**Str**:</u> Suppose we are given a cocartesian fibration $p : \mathcal{E} \to \mathcal{B}$. We want a functor $F : \mathcal{B} \to \mathrm{Cat}_{\infty}$. One thing one can always do to obtain a ∞-Category for every object in $\mathcal{B}$, is to take fibers over p. This leaves us with a mapping $b \mapsto \mathcal{E}_b$.
How do we come up with a mapping $f : b \to b' \mapsto \mathcal{E}_b \to \mathcal{E}_{b'}$? Note that on the righthandside we have to provide a functor of simplicial sets!
Let's see what we can do on object level first.
Suppose we are given a $x \in (\mathcal{E}_b)_0$, then by Definition x maps to b under p. Hence, finding a morphism f_x target $x' \in \mathcal{E}_{b'}$ under f_x, is the exact same thing as solving the following lifting problem:



Phew! Good thing that p is a cocartesian fibration, which means we can solve exactly such kind of lifting problems. And not only that. The arrow f_x, that we get, is a p-cocartesian morphims. Next, we would like to try ourselves on morphisms. Suppose we are given an $\alpha : x \to y \in (\mathcal{E}_b)_1$. Above's procedure provides us with $x', y' \in \mathcal{E}_{b'}$ and morphisms $f_x : x \to x'$, $f_y : y \to y'$. Now, specifiying a map $\hat{\alpha} : x' \to y'$ is the exact same thing as solving the following lifting problem:



Again, the setting is to our favor. This time we use that f_x is a p-cocartesian morphism and as such, all lifting problems of the above kind can be solved.

Claim: Given any $f : b \to b'$, there exists a functor $f_! : \mathcal{E}_b \to \mathcal{E}_{b'}$ extending the sketched procedure.
Recall, that any trivial fibration admits a section.

As observed in 2.8, $ev_0 : \mathrm{Fun}^{mcc}_f((\Delta^1)^\sharp, \mathcal{E}) \to \mathcal{E}_b$ is a trivial fibration, thus there exists a section ev_0^{-1}. Let $i : \{1\} \to \Delta^1$ be the obvious inclusion. We can thus define the functor

$$f_! : \mathcal{E}_b \xrightarrow{ev_0^{-1}} \mathrm{Fun}^{mcc}_f((\Delta^1)^\sharp, \mathcal{E}) \xrightarrow{ev_1} \mathrm{Fun}^{mcc}_{fi}((\Delta^0)^\sharp, \mathcal{E}) \cong \mathcal{E}'_b$$

Since $\mathrm{Fun}^{mcc}((\Delta^1)^\sharp, \mathcal{E}) \subset \mathrm{Fun}(\Delta^1, \mathcal{E})$ is a inclusion, there exists a unique functor $\phi : \mathcal{E}_b \times \Delta^1 \to \mathcal{E}$ by the usual adjunction for mapping objects. This functor has the following properties:
1. $f_!$ corresponds to $\phi|_{\mathcal{E}_b \times \{1\}}$, because $f_!$ was defined as $ev_1 \circ ev_0^{-1}$ and $id \cong \phi|_{\mathcal{E}_b \times \{0\}}$.
2. By a similar argument as in the Prop 2.7, ϕ fits into a commuting diagram:

$$\begin{tikzcd}
\mathcal{E}_b \times \Delta^1 \arrow[r, "\phi"] \arrow[d] & \mathcal{E} \arrow[d, "p"] \\
\Delta^1 \arrow[r, "f"] & \mathcal{B}
\end{tikzcd}$$

3. By naturality of the adjunction, picking an element $\Delta^0 \xrightarrow{z} \mathcal{E}_b$, yields a morphism $\Delta^1 \cong \Delta^0 \times \Delta^1 \to \mathcal{E}_b \times \Delta^1 \xrightarrow{\phi} \mathcal{E}$. By 1. this morphism has source z, target $f_!(z)$, and even is a p-cocartesian lift, since $f_!$ factors through $\mathrm{Fun}^{mcc}((\Delta^1)^\sharp, \mathcal{E})$.

Now, 2. implies, that $f_!$ exactly coincides on objects of $\mathcal{E}_b$ with the functor sketched above. Whilst 3. tells us, that the same is true for morphisms, as p-cocartesian lifts are unique up to contractible choice. We have therefore proved the claim.

<u>**Un:**</u> Let's examine, what the Unstraightening Functor looks like on objects. By this means, we would like to construct a cocartesian fibration from a given functor $F : \mathcal{B} \to \mathrm{Cat}_\infty$. As I will point out now, this can be done in a general way.
For this, we define $\mathrm{Cat}_{\infty *//} \subset \mathrm{Fun}(\Delta^1, \mathrm{Cat}_\infty)$ as the full subcategory of functors of the form ${}^{\mathcal{C}}/_x \to \mathcal{C}$ for $x : \Delta^0 \to \mathcal{C}$. Also, let $t : \mathrm{Cat}_{\infty *//} \to \mathrm{Fun}(\Delta^1, \mathrm{Cat}_\infty) \overset{ev_1}{\to} \mathrm{Cat}_\infty$ the obvious projection to the target. Now, I'll have to invoke a fact, appearing in [CH22], which Markus pointed out to me.

Fact:[CH22] t is a cocartesian fibration.

Coming across this fact, one has to note, that Haugseng and Chu do not aim for a proof of the straightening equivalence in this paper. In fact, they assume the straighenting equivalence to hold, in order to obtain this result and there probably isn't a way to get there without this assumption.

Given this fact, it is quite easy to construct $\mathbf{Un}_0$ i.e $\mathbf{Un}$ on object level. For the given functor $F : \mathcal{B} \to \mathrm{Cat}_\infty$, we choose $\mathcal{E}_F$ to be the pullback:

$$\begin{tikzcd} \mathcal{E}_F \arrow[r] \arrow[d] & \mathrm{Cat}_{\infty *//} \arrow[d] \\ \mathcal{B} \arrow[r, "F"] & \mathrm{Cat}_\infty \end{tikzcd}$$

If you have already done [Lan21]Exercise 117, you know that a pullback of a cocartesian fibrations along any map is again a cocartesian fibration. If you didn't, this exercise is solved by simply invoking the definitions in terms of lifting properties. We thus define $\mathbf{Un}(F) := \mathcal{E}_F \to \mathcal{B}$.

NB: Suppose Lurie's straightening equivalence holds. Then one part of this equivalence tells us exactly that every cocartesian fibration over $\mathcal{B}$ arises by pulling back from $t : \mathrm{Cat}_{\infty *//} \to \mathrm{Cat}_\infty$ up to equivalence. It is therefore justified to call $t : \mathrm{Cat}_{\infty *//} \to \mathrm{Cat}_\infty$ the universal cocartesian fibration.

4 Corollaries of Straightening-Unstraightening

For the rest of this talk we assume that the Straightening-Unstraightening Theorem 1.2 holds.
We are now able to deduce some handy corollaries from this Theorem.

Corollary 4.1
$\mathrm{LFib}(\mathcal{B}) \simeq \mathrm{Fun}(\mathcal{B}, \mathrm{Spc})$

<u>Proof:</u> Recall, that we have defined Spc:=N(CW), where CW is the simplicial 1-category of CW complexes. We have also seen, that Spc is equivalent to the ∞-category Grpd_∞ of ∞-groupoids. In the last talk we also proved, that any cocartesian fibration $p : \mathcal{E} \to \mathcal{B}$ is a left fibration if and only if the fiber $\mathcal{E}_b$ is a ∞-groupoid for every $b \in \mathcal{B}$. We thus conclude by our previous observation of the construction of $\mathbf{Str}$, that $\mathbf{Str}|_{\mathrm{LFib}(\mathcal{B})}$ lands in $\mathrm{Fun}(\mathcal{B}, \mathrm{Spc})$.
To deduct the claimed statement, let $F : \mathcal{B} \to \mathrm{Spc} \simeq \mathrm{Grpd}_\infty \subset \mathrm{Cat}_\infty$ be a functor. Then, by Theorem 1.2

$$\forall b \in \mathcal{B} : \quad \mathrm{Un}(F)_b = \mathrm{Str}(\mathrm{Un}(F))(b) \simeq F(b) \in \mathrm{Grpd}_\infty$$

Hence, $\mathrm{Un}(F)$ is a left fibration. $\square$.

Corollary 4.2
Let $\mathcal{B} \in \mathrm{Spc}$, *then* ${}^{\mathrm{Spc}}/_{\mathcal{B}} \simeq \mathrm{Fun}(\mathcal{B}, \mathrm{Spc})$.

Proof: By the previous Corollary, we just have to show, that ${}^{\mathrm{Spc}}/_{\mathcal{B}} \simeq \mathrm{LFib}(\mathcal{B})$.
Therefore, let $p : \mathcal{E} \to \mathcal{B}$ be a left fibration. In order for p to be a Kan fibration, we would have to solve the following lifting problem, where we additionally highlight the last edge by ϕ:

$$\begin{tikzcd} \Delta^{n-1,n} \arrow[r, hook] \arrow[rr, bend left, "\phi"] & \Lambda^n_n \arrow[r] \arrow[d, hook] & \mathcal{E} \arrow[d] \\ & \Delta^n \arrow[r] \arrow[ur, dashed, "\exists"] & \mathcal{B} \end{tikzcd}$$

We have seen, that p is conservative, since it is a left fibration. Now, $p(\phi)$ is an equivalence, which implies that ϕ is an equivalence. Hence, we conclude that the lifting problem above admits a solution by the Joyal-Lifting-Theorem (see [Lan21]).
This tells us, that $\mathrm{LFib}(\mathcal{B}) \subset {}^{\mathrm{Grpd}_\infty}/_{\mathcal{B}}$ is the full subcategory of Kan fibrations over $\mathcal{B}$. By the small object argument wrt. the Kan-Quillen-Model, each element of ${}^{\mathrm{Grpd}_\infty}/_{\mathcal{B}}$ is equivalent to a Kan fibration. Therefore, the inclusion of ∞-categories above is also essentially surjective, thus an equivalence. In order to fully prove the statement of this corollary, we still have to show, that ${}^{\mathrm{Grpd}_\infty}/_{\mathcal{B}} \simeq {}^{\mathrm{Spc}}/_{\mathcal{B}}$.
This is an instance of following, general fact.

Fact: Let $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}_\infty$, $K \in \mathbf{sSet}$, then $K \overset{\phi}{\to} \mathcal{C} \simeq \mathcal{D}$ implies $\mathcal{C}_{/\phi} \simeq {}^{\mathcal{D}}/_{\phi}$.

Here is a quick sketch, why one might think that this is true.

Suppose in **sSet** we artificially add inverses for all Joyal-equivalences [3]. Let's call this category $\mathbf{sSet}[\mathrm{W}^{-}1]$. Then, for any $L \in \mathrm{h}\mathbf{sSet}[\mathrm{W}^{-}1]$, the maps $[L, \mathcal{C}_{/\phi}]$ up to homotopy, are given by maps $[L \star K, \mathcal{C}]$ up to homotpy, which restrict to ϕ on K. Since we have added inverses for equivalences, $[L \star K, \mathcal{C}] = [L \star K, \mathcal{D}]$. Therefore, maps $[L \star K, \mathcal{C}]$, which restrict to ϕ on K are again $[L, \mathcal{D}_{/\phi}]$. Now the fact follows by Yoneda.

Remark 4.3
This Corollary generalizes Lemma 2.28 of the Topology 3 Course[4]. The statement of this lemma was as follows:
Let $E \to B$ be a fibration in the category **Top** of *sufficently nice topological spaces.* Let $e \in E$ and $b = p(e)$. Then there exist functors ϕ_p, ψ_p making the following diagram commute:

$$\begin{tikzcd} \tau_{\leq 1}(E) \arrow[r, "\psi_p"] \arrow[d] & h\mathbf{Top}_* \arrow[d] \\ \tau_{\leq 1}(B) \arrow[r, "\phi_p"] & h\mathbf{Top} \end{tikzcd}$$

Recall, that we obtained these functors ϕ_p, ψ_p by homotopy lifting. The functor ϕ_p for example was of the form

$$\begin{aligned} \tau_{\leq 1}(B) &\to h\mathbf{Top} \\ b &\mapsto E_b \\ b \overset{\gamma}{\to} b' &\mapsto h(-,1) : E_b \to E_{b'} \qquad\qquad \text{obtained from } h : E_b \times [0,1] \to E \end{aligned}$$

How do we obtain this Lemma from Cor 4.2 then?

[3]This process can be made precise in terms of model categories. Note that all Joyal-equivalences, inner-fibratons and inner-anodyne maps build the so called Joyal-model-category.
[4]see `https://www.mathematik.uni-muenchen.de/~land/Topo3.php`

By cellular approximation h**Top** $\cong$ hSpc $\cong$ hGrpd$_\infty$. Define Grpd$_{\infty *//}$ as the full subcategory spanned by functors $\mathcal{C}_{x/} \to \mathcal{C}$ for $\mathcal{C} \in$ Grpd$_\infty$. One then observes that this ∞-category is precisely the pullback of $t : \mathrm{Cat}_{\infty *//} \to \mathrm{Cat}_\infty$ along $\mathrm{Grpd}_\infty \subset \mathrm{Cat}_\infty$.

Given the intuiton for the fact above, it is reasonable to suppose that a functor $F : ({}^{\mathcal{C}}/_{x} \to \mathcal{C}) \to ({}^{\mathcal{D}}/_{y})$ in Grpd$_{\infty *//}$ if and only if the induced functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence and $F(x) \simeq y$ in $\mathcal{D}$.

Thus, we observe that $h\mathrm{Grpd}_{\infty *//} \cong h\mathbf{Top}$ and this iso commutes with the projections down to hGrpd$_\infty$ and h**Top**.
Having endured these technicalities, we define $\mathcal{B} := Sing(B)$ and $\mathcal{E} := Sing(E)$. Clearly, the induced morphism $Sing(p) : \mathcal{E} \to \mathcal{B}$ is a Kan-fibration. Now, set $\Phi_p := \mathbf{Str}(Sing(p))$. As examined earlier, this functor has the following form on the object and morphism level:

$$\begin{aligned}
\Phi_p : \mathcal{B} \to & \mathrm{Grp}_\infty \\
b \mapsto & \mathcal{E}_b = Sing(E_b) \\
b \xrightarrow{\gamma} b' \mapsto & \gamma_! : \mathcal{E}_b \to \mathcal{E}_{b'} \text{ s.t. } \gamma_! = h|_{\mathcal{E}\times\{1\}} \text{ of a homotopy } h : \mathcal{E}_b \times \Delta^1 \to \mathcal{E}
\end{aligned}$$

Thus, we conclude that $\phi_p = h\Phi_p$!
By using **Un** on Φ_p i.e. pulling back the universal cocartesian fibration along Φ_p, we obtain the following diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "\simeq"] \arrow[dr, "Sing(p)"'] & \mathcal{E}_{\Phi_p} \arrow[r, "\Psi_p"] \arrow[d] & \mathrm{Grpd}_{\infty *//} \arrow[d] \\
& \mathcal{B} \arrow[r, "\Phi_p"] & \mathrm{Grpd}_\infty
\end{tikzcd}$$

If we use the homotopy category functor on the whole diagram above, we now have two functors hΨ_p and ψ_p making the following diagram commute.

$$\begin{tikzcd}
h(\mathcal{E}) = \tau_{\leq 1}(E) \arrow[r, "\psi_p"] \arrow[r, bend left, "h\Psi_p"] \arrow[d, "p"] & h\mathbf{Top}_* \arrow[d] \\
h(\mathcal{B}) = \tau_{\leq 1}(B) \arrow[r, "\phi_p"] & h\mathbf{Top}
\end{tikzcd}$$

One then realizes, that Ψ_p sends all objects of $\mathcal{E}_{\Phi_p}$ to functors of the form ${}^{\mathcal{E}_b}/_{z} \to \mathcal{E}_b$ for $b \in \mathcal{B}$ and $z \in \mathcal{E}_b$. Clearly, these functors each correspond to the pointed fiber E_b with basepoint z. Hence, the functors hΨ_p and ψ_p take isomorphic values on each object of h$\mathcal{E}$, which directly implies that h$\Psi_p \cong \psi_p$ as functors on 1-categories.

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